Asymptotic Stability of the Faraday Wave Problem

by

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Abstract

This thesis concerns the dynamics of Faraday waves, a layer of incompressible viscous fluid lying above a vertically oscillating rigid plane with an upper boundary given by a free surface. We consider the problem with gravity and both with and without surface tension for horizontally periodic flows. This problem gives rise to steadily oscillating solutions, and the main thrust of this paper is to study the asymptotic stability of these solutions in certain parameters regimes. More specifically, we prove that there exists a parameter regime of the amplitude and frequency of the oscillation in which sufficiently small perturbations of the equilibrium at time $t = 0$ give rise to global-in-time solutions that return to equilibrium exponentially quickly in the case of fixed surface tension and almost exponentially in the case of vanishing surface tension.

This thesis is based on joint work with David Altizio, Ian Tice, and Xinyu Wu.

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Chapter 1

Introduction

Consider the following physics experiment: we have a flat rigid surface with a layer of fluid on top, and we oscillate the surface vertically (Figure 1.1).

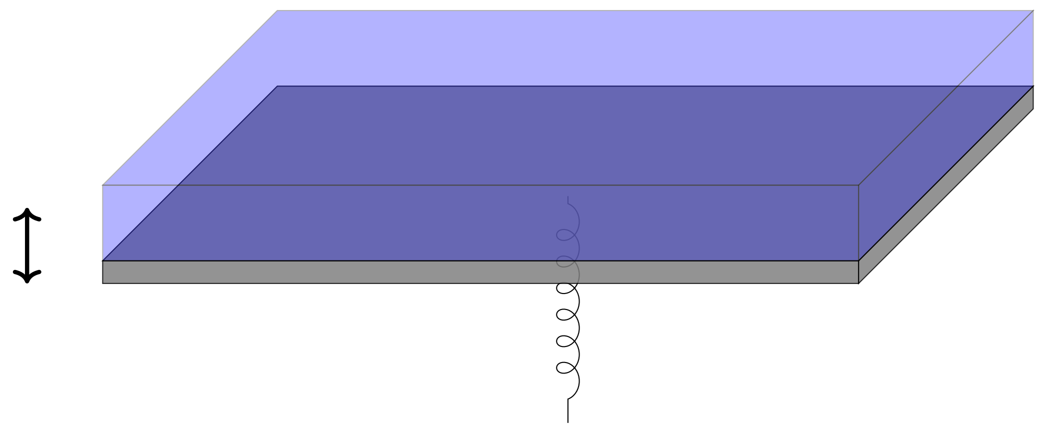

Figure 1.1: A layer of fluid evolves on a vertically oscillating rigid surface.

When the oscillation occurs above some threshold amplitude and frequency, the free surface of the fluid forms standing waves [Far31]. This phenomenon is known as *Faraday waves*, and the various fascinating patterns formed by these standing waves have been studied intensively, both experimentally and theoretically.

When the forcing oscillation occurs with a sufficiently small amplitude and frequency, the surface of the fluid remains flat. This is the parameter regime that we focus on in this work: we present a fully nonlinear analysis that shows that under some conditions, this solution is asymptotically stable.

In this chapter, we discuss preliminaries that will be essential for understanding our results and techniques. In section 1.1, we describe Faraday waves in more detail, especially about their equations of motion. In section 1.2, we recall the notion of stability for differential equations. Section 1.3 includes an overview of Sobolev spaces, which will be essential in understanding some of the more technical details of our results. Finally, we give a heuristic overview of how we approach the problem in section 1.4.

1.1 Faraday waves and their mathematical description

Since Faraday's discovery in 1831, Faraday waves have been studied intensively. On the theoretical end, the linearized problem has been analyzed both in the inviscid [BU54] and the viscous [KT94] cases to determine conditions for the onset of these surface waves. In particular, the inviscid case is known to be equivalent to the Mathieu equations [BU54]. Furthermore, a line of work has succeeded in explaining various surface wave patterns observed in experiments through weakly nonlinear analysis [WBVDW03, SR15]. Simulation studies also have achieved results that agree well with experiments in various settings [PJT09, Qad18]. To the best of our knowledge, there has been no fully nonlinear analysis of the Faraday wave problem.

Furthermore, Faraday waves have recently experienced a renewed interest ever since the experimental work of [CPFB05], which showed that Faraday waves coupled with water droplets can "walk" (Figure 1.2). These walking water droplets can further be coupled with other water

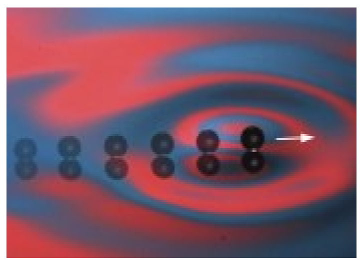

Figure 1.2: A "walking" water droplet [HQP+17].

droplets, and have been shown to exhibit behavior that has analogs to quantum mechanical phenomena. We refer the reader to the review of [Bus15] for more details on this exciting line of work.

We now introduce the equations that describe the Faraday wave problem. On the inside of our fluid, the interactions are described by the viscous incompressible Navier-Stokes equations in three dimensions, and at the boundary of our fluid, we have an oscillating rigid boundary on the bottom as well as a free boundary at the top.

1.1.1 The domain

In this work, we take our fluid to be horizontally periodic and thus we have no boundary at the "sides" of our fluid (e.g. the walls of a box in which the fluid resides). To model this, we introduce the horizontal cross section

$$\Sigma = (L_1\mathbb{T} \times L_2\mathbb{T}) \tag{1.1.1}$$

for horizontal periodicity parameters $L_1, L_2 > 0$ as in [Tic18]. Then, we take the free upper boundary to be specified by an unknown function $\eta : \Sigma \times [0, \infty) \to \mathbb{R}$ for each horizontal cell position $x' \in \Sigma$ and time t. In the setting of the Faraday wave problem, the bottom boundary is rigid and oscillates vertically. Typically, the oscillation is taken to be a single sinusoidal $A\cos(\omega t)$ with amplitude A and frequency ω, or an oscillation with multiple frequencies, as considered in [SR15]. However, we will generalize this setting and consider arbitrary oscillation profiles $f : \mathbb{T} \to [-1, 1]$ with sufficient regularity, so that the vertical component of the lower boundary at time t is given by $Af(\omega t) - b$, where $b > 0$ is a constant depth parameter.

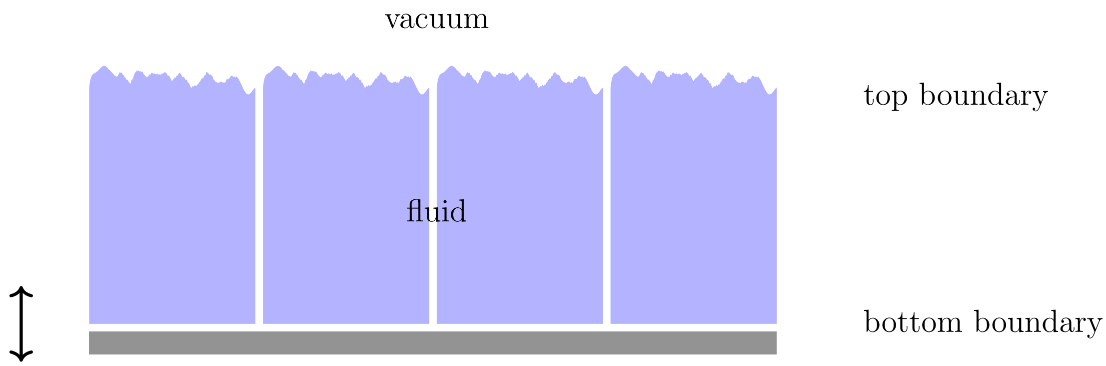

Figure 1.3: Side view of the top free boundary and bottom rigid oscillating boundary of a horizontally periodic fluid.

Thus, the moving fluid domain is modeled by the set

$$\Omega(t) = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : Af(\omega t) - b < x_3 < \eta(x', t)\}. \tag{1.1.2}$$

Note that the lower boundary of $\Omega(t)$ is the oscillating set

$$\Sigma_b(t) = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : x_3 = Af(\omega t) - b\} \tag{1.1.3}$$

while the moving upper surface is

$$\Sigma(t) = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : x_3 = \tilde{\eta}(x', t)\}. \tag{1.1.4}$$

1.1.2 Incompressible Navier-Stokes equations

For each time $t \geq 0$, let $\Omega(t) \subseteq \mathbb{R}^3$ be the open set in which our fluid resides. We take our fluid to be a viscous incompressible fluid, so we will use the incompressible Navier-Stokes equations, which gives

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u = F & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \end{cases} \tag{1.1.5}$$

where $u(\cdot, t) : \Omega(t) \to \mathbb{R}^3$ is the Eulerian velocity of the fluid, $p(\cdot, t) : \Omega(t) \to \mathbb{R}$ is the pressure, $\mu \geq 0$ is the viscosity, and $F(\cdot, t) : \Omega(t) \to \mathbb{R}^3$ is the external forcing term. In our case, the only external force we have is a uniform gravitational force field that points downwards with constant magnitude $g > 0$, i.e. $F = -ge_3$.

We briefly discuss the interpretation of these equations. The first equation is the balance of momentum equation and is analogous to Newton's third equation $F = ma$. The $\partial_t u + u \cdot \nabla u$ term corresponds to the acceleration of the fluid particles, suggested by the $\partial_t u$ term, the time derivative of the Eulerian velocity. The extra term $u \cdot \nabla u$ is due to the fact that we are using Eulerian coordinates, the coordinate system that fixes a point in space and tracks the movement of particles that pass through that one point, rather than Lagrangian coordinates, the coordinate system that fixes a fluid particle and tracks the movement of that one particle. The $\nabla p - \mu \Delta u$ term is the divergence of the Cauchy stress tensor $S = pI - \mu \mathbb{D}u$ (here, $\mathbb{D}u = Du + (Du)^\top$ is the symmetric gradient), and can be thought of as a force term that is introduced by the traction that the fluid particles experience when they flow past each other. The second equation is simply the requirement that the fluid is incompressible, i.e. that the Eulerian velocity never diverges or converges.

1.1.3 Boundary conditions

We now specify our boundary conditions. At the bottom rigid boundary, we require that the fluid doesn't slip, i.e. that the difference in the velocities of the fluid and the boundary is 0. Thus, we have

$$u = \frac{d}{dt}(Af(\omega t) - b)e_3 = A\omega f'(\omega t)e_3 \tag{1.1.6}$$

on $\Sigma_b(t)$. At the free boundary, we add two equations relating the function η specifying the upper boundary with the rest of our variables.

The first equation is the *kinematic transport equation* for η,

$$\partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3, \tag{1.1.7}$$

which states that the kinematics of a particle at the boundary η must match the kinematics specified by the Eulerian velocity. Now consider the outward-pointing unit normal vector ν on $\Sigma(t)$. We note that ν can be written in terms of the surface graph function η as

$$\nu = \frac{1}{\sqrt{1 + |\nabla \eta|^2}}(-\partial_1 \eta, -\partial_2 \eta, 1). \tag{1.1.8}$$

Thus, this equation can also be succinctly written as

$$\partial_t \eta = (u, \nu)\sqrt{1 + |\nabla \eta|^2}. \tag{1.1.9}$$

The second equation is the *balance of stress equation*,

$$(pI - \mu \mathbb{D}u)\nu = (P_{\text{ext}} - \sigma \mathfrak{H}(\eta))\nu, \tag{1.1.10}$$

where $\nu \in \mathbb{R}^3$ is the outward-pointing unit normal vector on $\Sigma(t)$, $P_{\text{ext}} \in \mathbb{R}$ is the constant pressure above the fluid, $\sigma > 0$ is the surface tension coefficient, and

$$\mathfrak{H}(\eta) = \operatorname{div}\left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}\right) \tag{1.1.11}$$

is (minus) twice the mean curature of $\Sigma(t)$. This states that the Cauchy stress tensor $S = pI - \mu \mathbb{D}u$ is balanced by the external pressure P_{ext} and the stress caused by surface tension, which increases as the curvature of the surface increases and acts in the opposite direction of the normal vector.

1.1.4 The PDE

Gathering the equations introduced above, we arrive at the following system of partial differential equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \tilde{p} - \mu \Delta u = -g e_3 & \text{in } \Omega(t) \\ \operatorname{div} u = 0 & \text{in } \Omega(t) \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma(t) \\ (pI - \mu \mathbb{D}u)\nu = (P_{\text{ext}} - \sigma \mathfrak{H}(\eta))\nu & \text{on } \Sigma(t) \\ u = A\omega f'(\omega t)e_3 & \text{on } \Sigma_b(t) \end{cases}. \tag{1.1.12}$$

The problem is augmented with initial data $\eta_0 : \tilde{\Sigma} \to (b + Af(0), \infty)$, which determines the initial domain Ω_0, as well as an initial velocity field $u_0 : \Omega_0 \to \mathbb{R}^3$. Note that the assumption $\eta_0 > -b + Af(0)$ on Σ means that Ω_0 is well-defined.

Furthermore, we will assume that the constant $b > 0$ is chosen so that the mass of the fluid in a single periodic cell, which we require to be conserved in time due to incompressibility, is given by

$$\mathcal{M} := b|\Sigma| = bL_1L_2. \tag{1.1.13}$$

Rewriting this condition in terms of $\tilde{\eta}$ gives

$$b|\Sigma| = \mathcal{M} = \int_\Sigma [\tilde{\eta}(x', t) - (Af(\omega t) - b)] \, dx' = b|\Sigma| + \int_\Sigma [\tilde{\eta}(x', t) - Af(\omega t)] \, dx' \tag{1.1.14}$$

or equivalently,

$$\int_\Sigma [\tilde{\eta}(x', t) - Af(\omega t)] \, dx' = 0. \tag{1.1.15}$$

1.2 Stability of differential equations

Now that we have introduced our PDE for the Faraday wave problem, we describe what it means to find asymptotically stable solutions to this PDE. We first introduce the notion of stability for ODEs before defining it for PDEs. Finally, we will see some examples of how to prove the asymptotic stability of differential equations using energy estimates and Gronwall's inequality. These techniques will be very similar in spirit to the approach that we take in proving our main result.

Consider the following general form for an ordinary differential equation (ODE) for $x : [0, \infty) \to \mathbb{R}^n$:

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0 \end{cases} \tag{1.2.1}$$

Then, an *equilibrium point* of the ODE is point that gives a solution that doesn't depend on time.

Definition 1 (Equilibrium point)**.** A point $x_0 \in \mathbb{R}^n$ is an *equilibrium point* if $f(x_0) = 0$.

Note that if x_0 is an equilibrium point, then the constant function $x(t) = x_0$ is a solution the ODE, since $\dot{x}(t) = 0 = f(x_0)$ will be satisfied for all time. We consider the problem of whether this solution is *stable* or not, that is, whether we can stay near this equilibrium point for all of time if we start sufficiently close to it.

Definition 2 (Stable equilibrium point)**.** An equilibrium point x_0 is *stable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x_0 - y_0| < \delta$ and $y : [0, \infty) \to \mathbb{R}^n$ is a solution to the ODE $\dot{y}(t) = f(y(t))$ with $y(0) = y_0$, then $|x_0 - y(t)| < \varepsilon$ for all $t \geq 0$.

Furthermore, we introduce the notion of *attractive* equilibrium points, which are equilibrium points such that solutions starting sufficiently near the equilibrium tend to the equilibrium point in the limit.

Definition 3 (Attractive equilibrium point)**.** An equilibrium point x_0 is *attractive* if there exists $\delta > 0$ such that if $|x_0 - y_0| < \delta$ and $y : [0, \infty) \to \mathbb{R}^n$ is a solution to the ODE $\dot{y}(t) = f(y(t))$ with $y(0) = y_0$, then $\lim_{t \to \infty} |x_0 - y(t)| = 0$.

Finally, we can talk about *asymptotically stable* equilibrium points.

Definition 4 (Asymptotically stable equilibrium point)**.** An equilibrium point is *asymptotically stable* if it is both attractive and stable.

We can similarly talk about equilibria and their stability for partial differential equations (PDE). Consider a PDE for $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$ of the form

$$\begin{cases} \partial_t u(x,t) = f(u(x,t), Du(x,t), \ldots, D^k u(x,t)) \\ u(\cdot, 0) = u_0 \end{cases} \tag{1.2.2}$$

where $u_0 : \mathbb{R}^n \to \mathbb{R}^m$ is the initial data at time $t = 0$. We then introduce analogous definitions to ODEs for PDEs:

Definition 5 (Equilibrium solution)**.** A function $u_0 : \mathbb{R}^n \to \mathbb{R}^m$ is an *equilibrium solution* if $f(u_0, Du_0, \ldots, D^k u_0) = 0$.

Definition 6 (Stable equilibrium solution)**.** An equilibrium solution u_0 is *stable (with respect to a norm* $\|\cdot\|$*)* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|u_0 - v_0\| < \delta$ and $v : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$ is a solution to the PDE $\partial_t v = f(v, Dv, \ldots, D^k v)$ with $v(\cdot, 0) = v_0$, then $\|u_0 - v(\cdot, t)\| = \varepsilon$ for all $t \geq 0$.

Definition 7 (Attractive equilibrium solution)**.** An equilibrium solution u_0 is *attractive (with respect to a norm* $\|\cdot\|$*)* if there exists $\delta > 0$ such that if $\|u_0 - v_0\| < \delta$ and $v : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$ is a solution to the PDE $\partial_t v = f(v, Dv, \ldots, D^k v)$ with $v(\cdot, 0) = v_0$, then $\lim_{t\to\infty} \|u_0 - v(\cdot, t)\| = 0$.

Definition 8 (Asymptotically stable equilibrium solution)**.** An equilibrium solution is *asymptotically stable* if it is both attractive and stable.

Note that in the case of ODEs, we have used the Euclidean norm without loss of generality as our notion of distance, since all norms are equivalent on finite-dimensional spaces. However, in the case of PDEs, we are talking about norms on function spaces and thus we must specify the norm, for instance L^p norms and Sobolev norms.

1.2.1 Asymptotic stability via Gronwall's inequality

We now introduce a fundamental tool, Gronwall's inequality, to show the asymptotic stability of a very simple ODE. Similar ideas will be used for later examples as well as our main result of this thesis.

Theorem 1.2.1 (Gronwall's inequality)**.** *Suppose that* $z : [0, \infty) \to \mathbb{R}$ *is a solution to the differential inequality*

$$\begin{cases} \dot{z} \leq Cz \\ z(0) = z_0 \end{cases} \tag{1.2.3}$$

where $C \in \mathbb{R}$ *is a constant. Then,*

$$z(t) \leq z_0 \exp(Ct) \tag{1.2.4}$$

for all $t \geq 0$.

Proof. Consider the integrating factor $\exp(-Ct)$. Then, multiplying through by this factor and rearranging gives

$$\dot{z}(t)\exp(-Ct) - Cz(t)\exp(-Ct) = \frac{d}{dt}(z(t)\exp(-Ct)) \leq 0. \tag{1.2.5}$$

Then integrating from time 0 to t gives

$$z(t)\exp(-Ct) - z(0) \leq 0, \tag{1.2.6}$$

or by rearranging,

$$z(t) \leq z_0 \exp(Ct) \tag{1.2.7}$$

as desired. $\square$

We will now apply the above theorem to a simple example. Suppose that $x : [0, \infty) \to \mathbb{R}^n$ satisfies the ODE

$$\begin{cases} \dot{x} = -Cx \\ x(0) = x_0 \end{cases} \tag{1.2.8}$$

where $C > 0$. Then clearly, $x_0 = 0$ is the only equilibrium point. We will now show that this equilibrium is asymptotically stable. Let y_0 be an initial data point and let y be the corresponding solution. By multiplying both sides of the differential equation by y, we find that

$$\frac{d}{dt}\frac{|y|^2}{2} = y \cdot \dot{y} = -Cy \cdot y = -C|y|^2. \tag{1.2.9}$$

Then by Gronwall's inequality applied to the ODE

$$\begin{cases} \dot{z} \leq -2Cz \\ z(0) = |y_0|^2 \end{cases} \tag{1.2.10}$$

i.e. for $z(t) = |y(t)|^2$, we have that for any initial data y_0 and a corresponding solution y,

$$|y(t)|^2 \leq |y_0|^2 \exp(-2Ct). \tag{1.2.11}$$

Then, the equilibrium point $x_0 = 0$ is stable since for any $\varepsilon > 0$, we can set $\delta = \varepsilon$ so that

$$|x_0 - y_0| < \delta \implies |x_0 - y(t)| = |y(t)| \leq |y_0|\exp(-Ct) \leq |y_0| = |x_0 - y_0| < \delta = \varepsilon. \tag{1.2.12}$$

Furthermore, x_0 is attractive since for $\delta = 1$, we have

$$|x_0 - y_0| < \delta \implies \lim_{t\to\infty}|x_0 - y(t)| = \lim_{t\to\infty}|y(t)| \leq \lim_{t\to\infty}|y_0|\exp(-Ct) = 0. \tag{1.2.13}$$

Thus, x_0 is an asymptotically stable equilibrium point.

1.2.2 Asymptotic stability via energy estimates

We will now see that the technique of the above section also applies to showing the asymptotic stability of PDEs as well. Specifically, we will show that $v_0 = 0$ is an asymptotically stable equilibrium solution for the heat equation with respect to the the L^2 norm $\|\cdot\|_{L^2(\Omega)}$. The approach taken here will be very similar to techniques used in the our main result.

Suppose that $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a smooth, compactly supported solution to the partial differential equation

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u(\cdot, 0) = u_0 \end{cases} \tag{1.2.14}$$

Then, by multiplying by u on both sides and integrating over Ω, we have that

$$\int_\Omega u\partial_t u - u\Delta u = 0. \tag{1.2.15}$$

We may rewrite the first term as

$$\int_\Omega u\partial_t u = \int_\Omega \partial_t \frac{|u|^2}{2} = \frac{1}{2}\partial_t \int_\Omega |u|^2 = \frac{1}{2}\partial_t \|u\|^2_{L^2(\Omega)} \tag{1.2.16}$$

while we may rewrite the second term using integration by parts as

$$\begin{aligned} \int_\Omega -u\Delta u &= \int_\Omega -u \operatorname{div}(\nabla u) \\ &= \int_\Omega \nabla u \cdot \nabla u + \int_{\partial\Omega} u\nabla u \cdot \nu = \int_\Omega |\nabla u|^2 = \|\nabla u\|^2_{L^2(\Omega)}. \end{aligned} \tag{1.2.17}$$

Furthermore, by Poincaré's inequality, we have that

$$\|u\|^2_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)} \tag{1.2.18}$$

for some constant $C > 0$ only depending on Ω. Thus, we find that

$$\partial_t \|u\|^2_{L^2(\Omega)} + \frac{1}{C}\|u\|^2_{L^2(\Omega)} \leq \partial_t \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} = 0 \tag{1.2.19}$$

and thus by Gronwall's inequality again, we find that

$$\|u\|^2_{L^2(\Omega)} \leq \|u_0\|^2_{L^2(\Omega)} \exp\left(-\frac{t}{C}\right). \tag{1.2.20}$$

This implies the asymptotic stability of $v_0 = 0$ in exactly the same way as before.

1.3 Sobolev spaces

In the heat equation example from the previous section, Poincaré's inequality played a crucial role in making the proof work – by allowing us to bound the function u by its spatial derivative ∇u, we were able to invoke Gronwall's inequality and deduce exponential decay and thus asymptotic stability. However, in more complex PDEs, such as the PDE for the Faraday wave problem, we will have much more complicated interactions with which we need to make the same scheme work. Various time and space derivatives, both inside the domain and on the boundary, all interact with each other, and a simple Poincaré estimate will likely not be enough to make a similar proof work. This motivates the use of *Sobolev spaces*, a space of functions that have nice properties that equip us with a variety of estimates that will allow us to relate all the parts of the PDE to each other. Furthermore, although we will not discuss this, the functional analytic properties of this space turn out to be essential in constructing solutions.

In order to define Sobolev spaces, we must first weaken our notion of functions and derivatives. This weakening will introduce enough flexibility for us to prove estimates that we were previously unable to.

1.3.1 Distributions

We start off by defining *distributions*, which generalize the notion of functions. Rather than defining functions by their action on points, we define their action on smooth functions of compact support, which we call *test functions*.

Definition 9 (Test functions)**.** We define the space of *test functions* $\mathscr{D}(\Omega;\mathbb{R})$ as the set $C_c^\infty(\Omega;\mathbb{R})$ endowed with the sequential topology such that $\varphi_k \to \varphi$ in $\mathscr{D}(\Omega;\mathbb{R})$ if there exists a compact set K such that $\operatorname{supp}(\varphi_k), \operatorname{supp}(\varphi) \subseteq K$ for all $k \geq \ell$ for some ℓ and $\partial^\alpha(\varphi_k - \varphi) \to 0$ uniformly on K.

Once we have defined test functions, distributions are just continuous linear functionals on test functions.

Definition 10 (Distributions)**.** We define the space of *distributions* $\mathscr{D}'(\Omega;\mathbb{R})$ as the set of linear functionals $T : \mathscr{D}(\Omega;\mathbb{R}) \to \mathbb{R}$ such that $T(\varphi_k) \to T(\varphi)$ for all $\varphi_k \to \varphi$ in $\mathscr{D}(\Omega;\mathbb{R})$.

To see that distributions generalize functions, let $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$. Then it's easy to see that

$$T_f(\varphi) = \int_\Omega f\varphi \tag{1.3.1}$$

defines a distribution.

In order to study distributional solutions to PDEs, we need to define what it means to take a derivative a distribution.

Definition 11 (Distributional derivative)**.** Let $T \in \mathscr{D}'(\Omega;\mathbb{R})$ and $\alpha \in \mathbb{N}^n$. Then, we define the *distributional* α*th partial derivative of* T, written $\partial^\alpha T$, via

$$\partial^\alpha T(\varphi) = (-1)^{|\alpha|}\partial^\alpha \varphi. \tag{1.3.2}$$

1.3.2 Weak derivatives and Sobolev spaces

Note that by weakening the notion of functions and derivatives in the previous section, all "functions" have now become infinitely differentiable. In order to restore a notion of regularity, we introduce weak derivatives. Weakly differentiable functions are functions whose distributional derivatives are functions, rather than arbitrary distributions. This turns out to achieve the right amount of regularity restrictions on a function for our purposes.

Definition 12 (Weak derivative)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open, let $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$, and let $\alpha \in \mathbb{N}^n$. Then g is the *weak* α*th partial derivative of* f if for all $\varphi \in C_c^\infty(\Omega;\mathbb{R})$,

$$\int_\Omega g\varphi = (-1)^{|\alpha|}\int_\Omega f\partial^\alpha \varphi. \tag{1.3.3}$$

With the notion of weak derivatives in hand, we introduce Sobolev spaces and Sobolev-Hilbert spaces. From here on, the partial derivatives $\partial^\alpha f$ for a function f will refer to weak partial derivatives rather than classical partial derivatives or distributional partial derivatives.

Definition 13 (L^p-based Sobolev space)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open and let $1 \leq p \leq \infty$. Then, the *L^p-based Sobolev space of order k* is the space

$$W^{k,p}(\Omega;\mathbb{R}) = \{f \in L^p(\Omega) : \forall \alpha \in \mathbb{N}^n \text{ s.t. } |\alpha| \leq k,\, \partial^\alpha f \in L^p(\Omega;\mathbb{R})\}. \tag{1.3.4}$$

We endow $W^{k,p}(\Omega;\mathbb{R})$ with the norm

$$\|f\|_{W^{k,p}(\Omega)} \coloneqq \left(\sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(\Omega)}^p\right)^{1/p} \tag{1.3.5}$$

for $1 \leq p < \infty$ and

$$\|f\|_{W^{k,\infty}(\Omega)} \coloneqq \max_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^\infty(\Omega)}. \tag{1.3.6}$$

Definition 14 (L^2-based Sobolev-Hilbert space)**.** Let $\Omega \subseteq \mathbb{R}^n$. Then, the L^2-based Sobolev-Hilbert space of order k is the space $H^k(\Omega;\mathbb{R}) \coloneqq W^{k,2}(\Omega;\mathbb{R})$ endowed with the inner product

$$\langle f, g\rangle_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega \langle \partial^\alpha f, \partial^\alpha g\rangle. \tag{1.3.7}$$

1.3.3 Tempered distributions and fractional Sobolev-Hilbert spaces

We will now make use of Fourier transforms in order to define fractional L^2-based Sobolev-Hilbert spaces. In order to do this, we will need to go back to distributions, rather than functions. However, in return, we get the ability take fractional, and even negative, derivatives.

We first need to introduce the Schwartz class, which will serve as a space of test functions for our tempered distributions, which are the distributions on which we will be able to take the Fourier transform.

Definition 15 (Schwarz class)**.** For $\alpha \in \mathbb{N}^n$, $m \in \mathbb{N}$, and $f \in C^\infty(\mathbb{R}^n;\mathbb{C})$, let

$$[f]_{\alpha,m} \coloneqq \sup_{x\in\mathbb{R}^n} (1 + |x|)^m |\partial^\alpha f|. \tag{1.3.8}$$

Then, the *Schwarz class* is the space of functions

$$\mathscr{S}(\mathbb{R}^n;\mathbb{C}) \coloneqq \{f \in C^\infty(\mathbb{R}^n;\mathbb{C}) : [f]_{\alpha,m} < \infty \text{ for all } \alpha \in \mathbb{N}^n \text{ and } m \in \mathbb{N}\}. \tag{1.3.9}$$

We endow the Schwarz class with the metric

$$d(f,g) = \sum_{m\in\mathbb{N}} \sum_{\alpha\in\mathbb{N}^n} \frac{1}{2^{m+|\alpha|}} \frac{[f-g]_{\alpha,m}}{1 + [f-g]_{\alpha,m}}. \tag{1.3.10}$$

The tempered distributions $\mathscr{S}'(\mathbb{R}^n;\mathbb{C})$ are now continuous linear functionals on the Schwarz class.

Definition 16 (Tempered distributions)**.** We define the space of *tempered distributions* $\mathscr{S}'(\Omega;\mathbb{C})$ as the set of linear functions $T : \mathscr{S}(\Omega;\mathbb{C}) \to \mathbb{R}^n$ such that $T(\varphi_k) \to T(\varphi)$ for all $\varphi_k \to \varphi$ in $\mathscr{S}(\Omega;\mathbb{C})$.

Note that the tempered distributions are a subset of usual distributions. However, they are special since we can meaningfully define a Fourier transform of a tempered distribution as follows.

Definition 17 (Fourier transform of a tempered distribution)**.** Let $T \in \mathscr{S}'(\mathbb{R}^n;\mathbb{C})$ be a tempered distribution. Then, we define the Fourier transform $\hat{T}$ of T via

$$\hat{T}(\varphi) = T(\hat{\varphi}). \tag{1.3.11}$$

Fourier transforms then allow us to define fractional Sobolev-Hilbert spaces.

Definition 18 (Fractional Sobolev-Hilbert space)**.** Let $s \in \mathbb{R}$. We define the L^2-based Sobolev-Hilbert space of order s via

$$H^s(\mathbb{R}^n;\mathbb{C}) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n;\mathbb{C}) : \hat{f} \in L^1_{\mathrm{loc}} \text{ and } \|f\|_{H^s} < \infty \right\} \tag{1.3.12}$$

where

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s \left| \hat{f}(\xi) \right|^2 \right)^{1/2}. \tag{1.3.13}$$

We endow $H^s(\mathbb{R}^n;\mathbb{C})$ with the inner product

$$\langle f, g \rangle_{H^s} := \int_{\mathbb{R}^n} (1+|\xi|^2) \hat{f}(\xi) \overline{g(\xi)} \, d\xi. \tag{1.3.14}$$

1.3.4 Traces

We will finally see an example of how all the work we put in developing fractional Sobolev-Hilbert spaces comes to fruition. Recall that in the Faraday wave problem PDE, we had differential equations specifying the Eulerian velocity u on the boundary of our domain. The restriction of a Sobolev function to its boundary is known as its trace, and it turns out that we can develop meaningful bounds on the Sobolev norm of the trace.

For this example, we define traces onto subspaces. The proof is omitted for simplicity. Rather, the focus is to see the kinds of estimates granted to us by fractional Sobolev-Hilbert spaces.

Let $1 \leq m < n$. We first define the trace operator restricted to Schwarz functions, i.e. $T : \mathscr{S}(\mathbb{R}^n;\mathbb{C}) \to \mathscr{S}(\mathbb{R}^m;\mathbb{C})$ via

$$Tf(x) = f(x_1, \ldots, x_m, 0, \ldots, 0). \tag{1.3.15}$$

Then, the following holds:

Theorem 1.3.1 (Traces onto subspaces)**.** *Let $s > (n-m)/2$. Then, T extends to a bounded linear map $T : H^s(\mathbb{R}^n;\mathbb{C}) \to H^{s-(n-m)/2}(\mathbb{R}^m;\mathbb{C})$, that is, we have the estimate*

$$\|Tf\|_{H^{s-(n-m)/2}} \lesssim \|f\|_{H^s}. \tag{1.3.16}$$

Note that in the case of our Faraday wave problem, one example application is the restriction of a three dimensional function, i.e. $n = 3$, to its boundary in two dimensions, i.e. $m = 2$. Thus, we bound $s - 1/2$ derivatives of the boundary by s derivatives in the bulk, so we really had to work with fractional derivatives. We will see these kinds of bounds used repeatedly in the proof of our main result.

1.4 Heuristic overview

Finally, we present a simplified overview of our main result.

1.4.1 Linearization

In the proof of our results, we make various reparametrizations of the PDE so that the domain is stationary and flattened, i.e. both the top and bottom boundaries are just $\mathbb{T}^2$. On top of this, we will *linearize* the PDE in this overview. Informally, linearization can be thought of as the Taylor expansion of the PDE around an equilibrium solution and throwing away all but the first order term. When the lower order terms are sufficiently small, then this gives us a good approximate picture of the behavior of the PDE with a much simpler system.

Recall the original PDE given in equation (1.1.12). Upon reparametrization and linearization, the linear PDE we get is

$$\begin{cases} \partial_t u + \nabla p - \mu \Delta u = 0 & \Omega, \\ \operatorname{div} u = 0 & \Omega, \\ S\nu = -\left[\sigma \Delta \eta - \left(g + A\omega^2 f''(\omega t)\right) \eta\right] \nu & \Sigma, \\ \partial_t \eta = u_3 & \Sigma, \\ u = 0 & \Sigma_b \end{cases} \tag{1.4.1}$$

where Ω is the domain, Σ is the top boundary, and Σ_b is the bottom boundary.

1.4.2 Energy-dissipation estimates

With the simplified PDE in hand, we can now proceed very similarly to what we did in section 1.2.2. We first take the first equation, multiply by u, and integrate over Ω just as we did in the heat equation. Recall that

$$\int_\Omega \partial_t u \cdot u = \partial_t \int_\Omega \frac{|u|^2}{2}. \tag{1.4.2}$$

We also compute

$$\begin{aligned} \int_\Omega (\nabla p - \mu \Delta u) \cdot u &= \int_\Omega \operatorname{div}(pI - \mu \mathbb{D} u) \cdot u \\ &= \int_{\partial\Omega} (pI - \mu \mathbb{D} u)\nu \cdot u - \int_\Omega (pI - \mu \mathbb{D} u) : \nabla u \\ &= \int_{\partial\Omega} S\nu \cdot u - \int_\Omega (p \operatorname{div}(u) - \mu |\nabla u|^2) \\ &= \int_\Sigma S\nu \cdot u + \int_{\Sigma_b} S\nu \cdot u + \mu \int_\Omega |\nabla u|^2 \\ &= \int_\Sigma S\nu \cdot u + \mu \int_\Omega |\nabla u|^2 \end{aligned} \tag{1.4.3}$$

using $\operatorname{div} u = 0$ in Ω and $u = 0$ on Σ_b. Now,

$$\begin{aligned}\int_\Sigma S\nu \cdot u &= \int_\Sigma -\left[\sigma\Delta\eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right](u\cdot\nu) \\ &= \int_{\mathbb{T}^2} -\left[\sigma\Delta\eta - \left(g + A\omega^2 f''(\omega t)\right)\eta\right]\partial_t\eta \\ &= \partial_t\left[\int_{\mathbb{T}^2}\frac{\sigma|\nabla\eta|^2}{2} + \frac{(g + A\omega^2 f''(\omega t))\,|\eta|^2}{2}\right] - \int_{\mathbb{T}^2}\frac{A\omega^3 f'''(\omega t)|\eta|^2}{2}.\end{aligned} \tag{1.4.4}$$

Thus, on sum, we find that

$$\partial_t\left(\int_\Omega \frac{|u|^2}{2} + \int_{\mathbb{T}^2}\frac{\sigma|\nabla\eta|^2}{2} + \frac{(g + A\omega^2 f''(\omega t))\,|\eta|^2}{2}\right) + \mu\int_\Omega|\mathbb{D}u| = \int_{\mathbb{T}^2}\frac{A\omega^3 f'''(\omega t)|\eta|^2}{2}. \tag{1.4.5}$$

Now consider defining the following energy, dissipation, and forcing terms:

$$\begin{aligned}\mathcal{E} &:= \int_\Omega \frac{|u|^2}{2} + \int_{\mathbb{T}^2}\frac{\sigma|\nabla\eta|^2}{2} + \frac{(g + A\omega^2 f''(\omega t))\,|\eta|^2}{2} \\ \mathcal{D} &:= \mu\int_\Omega|\mathbb{D}u| \\ \mathcal{F} &:= \int_{\mathbb{T}^2}\frac{A\omega^3 f'''(\omega t)|\eta|^2}{2}\end{aligned} \quad . \tag{1.4.6}$$

Suppose that we could show that $\lambda\mathcal{E} \le \mathcal{D}$ and that $\mathcal{F} \le \frac{\lambda}{2}\mathcal{E}$ for some constant $\lambda > 0$. Then, we would conclude that

$$\partial_t\mathcal{E} + \frac{\lambda}{2}\mathcal{E} \le 0 \tag{1.4.7}$$

and thus that $\mathcal{E}$ decays exponentially. Intuitively, the estimates above would show that the forcing term can be absorbed into the dissipation in a way that still makes Gronwall's inequality work. However, we won't quite be able to do this at this point, since the $\mathcal{D}$ term only controls u for now. In order to fix this, we will create more differential equations by taking spatial and temporal derivatives of the original equations and prove similar energy-dissipation estimates as we did above. Then by adding these estimates, we obtain a version of the energy-dissipation estimate where the energy, dissipation, and forcing terms contain many more derivative terms. Once we do this, we will have more terms to play with, and we will be able to make use of Sobolev estimates and elliptic regularity estimates to show the desired bounds of $\lambda\mathcal{E} \le \mathcal{D}$ and $\mathcal{F} \le \frac{\lambda}{2}\mathcal{E}$. This is the sketch for the linearized proof.

When we show the main result for the fully nonlinear problem, we will have to show that the nonlinearities that we just threw away in our linearization can also be absorbed into the various terms so that the basic framework presented above still holds.

Chapter 2

Recasting the PDE

We make several modifications to the problem formulation to simplify future analyses. First let $\tilde{u}$, $\tilde{p}$, and $\tilde{\eta}$ denote our u, p, η from before, so that we may reserve these variable names for the corresponding values after our recasting.

2.1 Reparametrization

2.1.1 Absorbing the gravity

The first trick we play is to redeclare $\tilde{p}$ to be

$$\tilde{p}_{\text{new}} := \tilde{p}_{\text{old}} + gx_3 - P_{\text{ext}}. \tag{2.1.1}$$

Then, $\nabla \tilde{p}_{\text{new}} = \nabla \tilde{p}_{\text{old}} + ge_3$, so the first and fourth equations of eq. (1.1.12) become

$$\begin{cases} \partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} - \mu \Delta \tilde{u} = 0 & \text{in } \tilde{\Omega}(t) \\ (\tilde{p} I - \mu \mathbb{D} \tilde{u})\nu = (-\sigma \mathfrak{H}(\tilde{\eta}) + g\tilde{\eta})\nu & \text{on } \tilde{\Sigma}(t) \end{cases} \tag{2.1.2}$$

2.1.2 Change of coordinates

Next, we make a Galilean change of coordinates. The above formulation of the problem is intuitive as an external observer, but it is more convenient to view the problem from the frame of the fluid itself and fix the moving lower boundary. As such, we employ the following change of coordinates:

$$\begin{aligned} \tilde{u}(x,t) &= \bar{u}(x', x_3 - Af(\omega t), t) + A\omega f'(\omega t) e_3 \\ \tilde{p}(x,t) &= \bar{p}(x', x_3 - Af(\omega t), t) \\ \tilde{\eta}(x',t) &= \bar{\eta}(x',t) + Af(\omega t) \end{aligned} \tag{2.1.3}$$

By plugging the above into eq. (1.1.12) with the modifications in eq. (2.1.2), we obtain the new set of equations (details in appendix A.1)

$$\begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{p} - \mu \Delta \bar{u} + A\omega^2 f''(\omega t) e_3 = 0 & \text{in } \Omega(t) \\ \operatorname{div} \bar{u} = 0 & \text{in } \Omega(t) \\ \partial_t \bar{\eta} + \bar{u}_1 \partial_1 \bar{\eta} + \bar{u}_2 \partial_2 \bar{\eta} = \bar{u}_3 & \text{on } \Sigma(t) \\ (\bar{p} I - \mu \mathbb{D} \bar{u})\nu = (-\sigma \mathfrak{H}(\bar{\eta}) + g\left(\bar{\eta} + Af(\omega t)\right))\nu & \text{on } \Sigma(t) \\ \bar{u} = 0 & \text{on } \Sigma_b \end{cases} \tag{2.1.4}$$

where

$$\begin{aligned} \Omega(t) &= \{x = (x', x_3) \in \Sigma \times \mathbb{R} : -b < x_3 < \bar\eta(x', t)\} \\ \Sigma(t) &= \{x = (x', x_3) \in \Sigma \times \mathbb{R} : x_3 = \bar\eta(x', t)\} \\ \Sigma_b &= \{x = (x', x_3) \in \Sigma \times \mathbb{R} : x_3 = -b\} \end{aligned} \tag{2.1.5}$$

are the new versions of the domains where the lower boundary is now unmoving and the upper boundary is now defined by the new graph function $\bar\eta$.

2.1.3 Absorbing the oscillation acceleration

Finally, we play the same trick as before to remove the $A\omega^2 f''(\omega t)e_3$ from the first equation. We define

$$\bar p_{\text{new}} \coloneqq \bar p_{\text{old}} + A\omega^2 f''(\omega t)x_3 - gAf(\omega t) \tag{2.1.6}$$

so that on sum, we have

$$\begin{cases} \partial_t \bar u + \bar u \cdot \nabla \bar u + \nabla \bar p - \mu \Delta \bar u = 0 & \text{in } \Omega(t) \\ \operatorname{div} \bar u = 0 & \text{in } \Omega(t) \\ \partial_t \bar\eta + \bar u_1 \partial_1 \bar\eta + \bar u_2 \partial_2 \bar\eta = \bar u_3 & \text{on } \Sigma(t) \\ (\bar p I - \mu \mathbb{D} \bar u)\nu = \left(-\sigma \mathfrak{H}(\bar\eta) + (g + A\omega^2 f''(\omega t))\,\bar\eta\right)\nu & \text{on } \Sigma(t) \\ \bar u = 0 & \text{on } \Sigma_b \end{cases} \tag{2.1.7}$$

2.1.4 Synthesis

In summary, we have made the following reparametrization:

$$\begin{aligned} \tilde u(x,t) &= \bar u(x', x_3 - Af(\omega t), t) + A\omega f'(\omega t)e_3 \\ \tilde p(x,t) &= \bar p(x', x_3 - Af(\omega t), t) + P_{\text{ext}} - \left(g + A\omega^2 f''(\omega t)\right)(x_3 - Af(\omega t)) \\ \tilde\eta(x', t) &= \bar\eta(x', t) + Af(\omega t) \end{aligned} \tag{2.1.8}$$

Note that condition in equation (1.1.15) now becomes

$$\int_\Sigma \bar\eta(x', t)\; dx' = 0 \text{ for all } t \geq 0. \tag{2.1.9}$$

Note that for sufficiently regular solutions to eq. (1.1.12), we have $\partial_t \bar\eta = \bar u \cdot \nu \sqrt{1 + (\partial_1 \bar\eta)^2 + (\partial_2 \bar\eta)^2}$, and hence

$$\frac{d}{dt} \int_\Sigma \bar\eta = \int_\Sigma \partial_t \bar\eta = \int_{\Sigma(t)} \bar u \cdot \nu = \int_{\Omega(t)} \operatorname{div} \bar u = 0. \tag{2.1.10}$$

Thus, equation (1.1.15) holds provided that the initial surface function satisfies the "zero average" condition

$$\frac{1}{L_1 L_2} \int_\Sigma \bar\eta_0 = 0, \tag{2.1.11}$$

which we henceforth assume. This is not a loss of generality: see the introduction of [GT13] for an explanation of how to obtain this condition via a coordinate shift.

2.2 Steady oscillating solution

Note that $\overline{U}(x,t) = 0, \overline{P}(x,t) = 0, \overline{H}(x,t) = 0$[1] is a solution to the reparameterized system eq. (2.1.7) when we set $\bar{u} = \overline{U}, \bar{p} = \overline{P}, \bar{\eta} = \overline{H}$. In the original system, this corresponds to the steady oscillation solution

$$\begin{aligned} \tilde{U}(x,t) &= A\omega f'(\omega t) e_3 \\ \tilde{P}(x,t) &= P_{\text{ext}} - \left(g + A\omega^2 f''(\omega t)\right)(x_3 - Af(\omega t)) \\ \tilde{H}(x,t) &= Af(\omega t) \end{aligned} \tag{2.2.1}$$

and it is easy to check that this indeed satisfies system eq. (1.1.12) – the first equation is satisfied since the pressure cancels out the $\partial_t \tilde{u}$ and the gravity term, while the fourth equation is satisfied since $x_3 = \eta = Af(\omega t)$ on $\Sigma(t)$ and so the pressure is just P_{ext}.

We will study the problem in this reparametrization, with the aim of showing that the above steady oscillation solution is asymptotically stable for some range of the parameters. In order to justify why we might expect such a stability result, consider the natural energy-dissipation equation associated with eq. (2.1.7):

$$\frac{d}{dt}\left(\int_{\Omega(t)} \frac{|\bar{u}|^2}{2} + \int_\Sigma \frac{|\bar{\eta}|^2}{2} + \sigma\sqrt{1+|\nabla\bar{\eta}|^2}\right) + \int_{\Omega(t)} \frac{\mu|\mathbb{D}\bar{u}|^2}{2} + (g + A\omega^2 f''(\omega t))\partial_t \int_\Sigma \frac{|\bar{\eta}|^2}{2} = 0. \tag{2.2.2}$$

To ensure that the dissipation is positive, we consider the regime where $\|f''\|_{L^\infty} \leq 1$, and $g - A\omega^2 > c$ for a positive constant c.

Therefore, if we can absorb the oscillation term into the energy and dissipation, then we can expect the stability of the system.

2.3 Reformulation in a flattened coordinate system

The moving domain $\Omega(t)$ is inconvenient for analysis, so we will reformulate the problem eq. (2.1.7) in the fixed equilibrium domain

$$\Omega = \{x = (x', x_3) \in \Sigma \times \mathbb{R} : -b < x_3 < 0\}. \tag{2.3.1}$$

We will think of Σ as the upper boundary of Ω and view $\bar{\eta}$ as a function on $\Sigma \times \mathbb{R}^+$. We then define

$$\hat{\eta} := \mathcal{P}\bar{\eta} \tag{2.3.2}$$

to be the harmonic extension of $\bar{\eta}$ into the lower half space as in section C.2. Then, we flatten the coordinate domain via the mapping $\Phi : \Omega \times \mathbb{R}^+ \to \Omega(t)$

$$\Phi(x,t) = \left(x_1, x_2, x_3 + \hat{\eta}(x',t)\left(1 + \frac{x_3}{b}\right)\right). \tag{2.3.3}$$

Note that $\Phi(\cdot,t)$ is smooth and extends to $\overline{\Omega}$ in such a way that $\Phi(\Sigma,t) = \Sigma(t)$ and $\Phi(\Sigma_b,t) = \Sigma_b$, i.e. Φ maps Σ to the free surface and keeps the lower surface fixed. We have

$$\nabla\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A & B & J \end{pmatrix}, \qquad \mathcal{A} := \left(\nabla\Phi^{-1}\right)^\top = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix} \tag{2.3.4}$$

[1] Here, H is a capital η.

where under the notational convenience $\tilde{b} = (1 + x_3/b)$ we have

$$A = \partial_1 \hat{\eta} \tilde{b}, \qquad B = \partial_2 \hat{\eta} \tilde{b}, \qquad J = \left(1 + \frac{\hat{\eta}}{b} + \partial_3 \hat{\eta} \tilde{b} \right), \qquad K = J^{-1}. \tag{2.3.5}$$

Note that $J = \det \nabla \Phi$ is the determinant of the transformation.

Using the matrix $\mathcal{A}$, we define a collection of $\mathcal{A}$-dependent differential operators. We define the differential operators $\nabla_{\mathcal{A}}$ and $\operatorname{div}_{\mathcal{A}}$ with their actions given by

$$(\nabla_{\mathcal{A}} f)_i := \mathcal{A}_{ij} \partial_j f, \qquad \operatorname{div}_{\mathcal{A}} X := \mathcal{A}_{ij} \partial_j X_i \tag{2.3.6}$$

for appropriate f and X. We extend $\operatorname{div}_{\mathcal{A}}$ to act on symmetric tensors in the usual way. Now write the change of coordinates as

$$\begin{aligned} u(x,t) &= \bar{u}(\Phi(x,t),t) \\ p(x,t) &= \bar{p}(\Phi(x,t),t) \\ \eta(x',t) &= \bar{\eta}(x',t). \end{aligned} \tag{2.3.7}$$

We then also write

$$(\mathbb{D}_{\mathcal{A}} u)_{ij} := \mathcal{A}_{ik} \partial_k u_j + \mathcal{A}_{jk} \partial_k u_i, \qquad S_{\mathcal{A}}(u,p) := pI - \mu \mathbb{D}_{\mathcal{A}} u, \tag{2.3.8}$$

and we define

$$\mathcal{N} := (-\partial_1 \bar{\eta}, -\partial_2 \bar{\eta}, 1) \tag{2.3.9}$$

for the non-unit normal to $\Sigma(t)$. In this new coordinate system, the new system of PDEs eq. (2.1.7) becomes the following system (details in section appendix A.2):

$$\begin{cases} \partial_t u - \partial_t \hat{\eta} \tilde{b} K \partial_3 u + u \cdot \nabla_{\mathcal{A}} u + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(u,p) = 0 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\ S_{\mathcal{A}}(u,p) \mathcal{N} = \left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t) \right) \eta \right) \mathcal{N} & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \end{cases}. \tag{2.3.10}$$

Chapter 3

Main results and discussion

3.1 Notation and definitions

In order to properly state our main results we must first introduce some notation and define various functionals that will be used throughout the paper. We begin with some notational conventions.

Einstein summation and constants: We will employ the Einstein convention of summing over repeated indices for vector and tensor operations. Throughout the paper $C > 0$ will denote a generic constant that can depend on Ω and its dimensions as well as on g, μ, and the oscillation profile f, but not on the parameters A and ω. Such constants are referred to as "universal", and they are allowed to change from one inequality to another. We employ the notation $a \lesssim b$ to mean that $a \le Cb$ for a universal constant $C > 0$.

Norms: We write $H^k(\Omega)$ with $k \ge 0$ and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for the usual L^2-based Sobolev spaces. In particular $H^0 = L^2$. In the interest of concision, we neglect to write $H^k(\Omega)$ or $H^k(\Sigma)$ in our norms and typically write only $\|\cdot\|_k$. The price we pay for this is some minor ambiguity in the set on which the norm is computed, but we mitigate potential confusion by always writing the space for the norm when traces are involved.

Multi-indices: We will write $\mathbb{N}^k$ for the usual set of multi-indices, where here we employ the convention that $0 \in \mathbb{N}$. For $\alpha \in \mathbb{N}^k$ we define the spatial differential operator $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_k^{\alpha_k}$. We will also write $\mathbb{N}^{1+k}$ for the set of space-time multi-indices

$$\mathbb{N}^{1+k} = \{(\alpha_0, \alpha_1, \dots, \alpha_k : \alpha_i \in \mathbb{N} \text{ for } 0 \le i \le k\}. \tag{3.1.1}$$

For a multi-index $\alpha \in \mathbb{N}^{1+k}$, we define the differential operator $\partial^\alpha = \partial_t^{\alpha_0} \partial_1^{\alpha_1} \dots \partial_k^{\alpha_k}$. Also, for a space-time multi-index $\alpha \in \mathbb{N}^{1+k}$ we use the parabolic counting scheme $|\alpha| = 2\alpha_0 + \alpha_1 + \dots + \alpha_k$.

Energy and dissipation functionals: Throughout the paper we will make frequent use of various energy and dissipation functionals, dependent on time. We define these now. The basic and full energy functionals, respectively, are defined as

$$\overline{\mathcal{E}_n^\sigma} \coloneqq \sum_{\substack{\alpha \in \mathbb{N}^{1+2} \\ |\alpha| \le 2n}} \|\partial^\alpha u\|_0^2 + g\|\partial^\alpha \eta\|_0^2 + \sigma \|\nabla \partial^\alpha \eta\|_0^2 \tag{3.1.2}$$

and

$$\mathcal{E}_n^\sigma := \overline{\mathcal{E}_n^\sigma} + \sum_{j=0}^{n} \|\partial_t^j u\|_{2n-2j}^2 + \sum_{j=0}^{n-1} \|\partial_t^j p\|_{2n-2j-1}^2 + \sigma \|\eta\|_{2n-2j+1}^2 + \|\eta\|_{2n}^2 + \sum_{j=1}^{n} \left\|\partial_t^j \eta\right\|_{2n-2j+3/2}^2. \tag{3.1.3}$$

The corresponding basic and full dissipation functionals are

$$\overline{\mathcal{D}}_n := \sum_{\substack{\alpha \in \mathbb{N}^{1+2} \\ |\alpha| \leq 2n}} \|\mathbb{D}\partial^\alpha u\|_0^2 \tag{3.1.4}$$

and

$$\begin{aligned} \mathcal{D}_n^\sigma := \overline{\mathcal{D}}_n &+ \sum_{j=0}^{n} \|\partial_t^j u\|_{2n-2j+1}^2 + \sum_{j=0}^{n-1} \left\|\partial_t^j p\right\|_{2n-2j}^2 + \sum_{j=0}^{n-1} \left(\left\|\partial_t^j \eta\right\|_{2n-2j-1/2}^2 + \sigma^2 \left\|\partial_t^j \eta\right\|_{2n-2j+3/2}^2 \right) \\ &+ \sum_{j=3}^{n+1} \left\|\partial_t^j \eta\right\|_{2n-2j+5/2}^2 + \|\partial_t \eta\|_{2n-1}^2 + \sigma^2 \|\partial_t \eta\|_{2n+1/2}^2 + \left\|\partial_t^2 \eta\right\|_{2n-2}^2 + \sigma^2 \left\|\partial_t^2 \eta\right\|_{2n-3/2}^2. \end{aligned} \tag{3.1.5}$$

In order to absorb the oscillation term into the energy or dissipation, here is where we need the assumption that $g - A\omega^2$ is bounded below by a constant.

We will also need to make frequent reference to two functionals that are not naturally of energy or dissipation type. We refer to these as

$$\mathcal{F}_n := \|\eta\|_{2n+1/2}^2 \tag{3.1.6}$$

and

$$\mathcal{K} := \|u\|_{C_b^2(\Omega)}^2 + \|u\|_{H^3(\Sigma)}^2 + \|p\|_{H^3(\Sigma)}^2 + \|\eta\|_{5/2}^2. \tag{3.1.7}$$

3.2 Local existence theory

The main content of this work is the a priori estimates that can be combined with local existence theory in order to construct local-in-time solutions to the PDE of equation (2.3.10). In this section, we only state the local existence theory that is needed to make this work without proof. Such omission is accepted in the literature [Tic18], and is justified by the abundance of similar types of proof schemes, for instance in [GT13, WTK14, Wu14, TW14, JTW16].

To state these local existence results, we need to introduce function spaces in which our solutions exist, as well as *compatibility conditions*, which give sufficient constraints on our initial data for constructing solutions using our a priori estimates. Our function spaces are the following:

$$\begin{aligned} {}_0H^1(\Omega) &= \left\{ v \in H^1(\Omega; \mathbb{R}^3) : v \mid_{\Sigma_b} = 0 \right\} \\ \mathcal{X}_T &= \left\{ u \in L^2([0,T]; {}_0H^1(\Omega)) : \operatorname{div}_{\mathcal{A}(t)} u(t) = 0 \text{ for a.e. } t \in [0,T] \right\} \end{aligned} \tag{3.2.1}$$

where the $\mathcal{A}(t)$ here is determined by the $\eta : \Sigma \times [0,T] \to \mathbb{R}$ coming from the solution. We refer the reader to [GT13, WTK14, Wu14, TW14, JTW16] for the compatibility conditions, as they are simple yet cumbersome to record.

When $\sigma > 0$ is fixed and positive, we have the following local existence result [WTK14]:

Theorem 3.2.1 (Local existence for fixed positive σ)**.** *Let $\sigma > 0$ be fixed and positive and let $n \geq 1$ be an integer and suppose that the initial data (u_0, η_0) satisfy*

$$\|u_0\|_{2n}^2 + \|\eta_0\|_{2n+1/2}^2 + \sigma\|\nabla\eta\|_{2n}^2 < \infty \tag{3.2.2}$$

as well as the natural compatibility conditions associated with n. Then there exist $0 < \delta_*, T_* < 1$ such that if

$$\|u_0\|_{2n}^2 + \|\eta_0\|_{2n+1/2}^2 + \sigma\|\nabla\eta\|_{2n}^2 \leq \delta_* \tag{3.2.3}$$

and $0 < T \leq T_*$, then there exists a unique triple (u, p, η) that achieves the initial data, solves (2.3.10), and obeys the estimates

$$\sup_{0\leq t\leq T}(\mathcal{E}_n^\sigma(t) + \mathcal{F}_n(t)) + \int_0^T \mathcal{D}_n^\sigma(t)\ dt + \left\|\partial_t^{n+1}u\right\|_{(\mathcal{X}_T)^*}^2 \lesssim \|u_0\|_{2n}^2 + \|\eta_0\|_{2n+1/2}^2 + \sigma\|\nabla\eta\|_{2n}^2. \tag{3.2.4}$$

We also consider the vanishing surface tension regime, in which we obtain the following result by requiring n to be larger [GT13, Wu14, TW14, JTW16]:

Theorem 3.2.2 (Local existence for vanishing σ)**.** *Let $n \geq 2$ be an integer and suppose that the initial data (u_0, η_0) satisfy*

$$\|u_0\|_{2n}^2 + \|\eta_0\|_{2n+1/2}^2 + \sigma\|\nabla\eta\|_{2n}^2 < \infty \tag{3.2.5}$$

as well as the natural compatibility conditions associated with n. Then there exist $0 < \delta_*, T_* < 1$ such that if

$$\|u_0\|_{2n}^2 + \|\eta_0\|_{2n+1/2}^2 + \sigma\|\nabla\eta\|_{2n}^2 \leq \delta_* \tag{3.2.6}$$

and $0 < T \leq T_*$, then there exists a unique triple (u, p, η) that achieves the initial data, solves (2.3.10), and obeys the estimates

$$\sup_{0\leq t\leq T}(\mathcal{E}_n^\sigma(t) + \mathcal{F}_n(t)) + \int_0^T \mathcal{D}_n^\sigma(t)\ dt + \left\|\partial_t^{n+1}u\right\|_{(\mathcal{X}_T)^*}^2 \lesssim \|u_0\|_{2n}^2 + \|\eta_0\|_{2n+1/2}^2 + \sigma\|\nabla\eta\|_{2n}^2. \tag{3.2.7}$$

3.3 Statement of main results

The main result of this paper is the global well-posedness of the problem and decay of solutions.

Theorem 3.3.1. *Suppose that initial data (u_0, η_0) satisfy $\mathcal{E}_1^\sigma(0) < \infty$ as well as the compatibility conditions of theorem 3.2.1. There exist constants $\gamma_0 \in (0, 1)$, $\kappa_0 \in (0, 1)$, and $0 < c < g$ such that if $\mathcal{E}_1^\sigma(0) \leq \kappa_0$, $A\omega^2 + A\omega^3 \leq \gamma_0$, and $g - A\omega^2 > c$, then there exists a unique solution (u, p, η) solving eq.* (2.3.10) *on the temporal interval $(0, \infty)$, achieves the initial data, and there exists constants $\lambda > 0$ and $C > 0$, depending on A, ω and σ such that the solution obeys the energy estimate*

$$\sup_{0\leq t\leq\infty} e^{\lambda t}\mathcal{E}_1^\sigma(t) + \int_0^\infty \mathcal{D}_1^\sigma(t)\, dt \leq C\mathcal{E}_1^\sigma(0). \tag{3.3.1}$$

Theorem 3.3.1 requires a fixed positive value of surface tension. Our next main result considers the cases $\sigma = 0$ and σ small but positive. We view the latter as the "vanishing surface tension" regime, as we will employ it to establish this limit. In these cases we work in a more complicated functional setting that changes depending on whether σ vanishes or not. We introduce this with the following functional, defined for any integer $N \geq 3$ and time $t \in [0, \infty]$:

$$\mathcal{G}_{2N}^{\sigma}(t) := \sup_{0 \leq r \leq t} \mathcal{E}_{2N}^{\sigma}(r) + \int_0^t \mathcal{D}_{2N}^{\sigma}(r)\, dr + \sup_{0 \leq r \leq t} (1+r)^{4N-8} \mathcal{E}_{N+2}^{\sigma}(r) + \sup_{0 \leq r \leq t} \frac{\mathcal{F}_{2N}(r)}{1+r}, \tag{3.3.2}$$

where here $\mathcal{E}_n^{\sigma}$, $\mathcal{D}_n^{\sigma}$, and $\mathcal{F}_n$ are defined by eq. (3.1.3), eq. (3.1.5), and eq. (3.1.6), respectively. Note that the condition $N \geq 3$ implies that $2N > N+2$ and that $4N - 8 > 0$.

We can now state our second main result.

Theorem 3.3.2. *Let Ω be given by eq.* (2.3.1)*, let $N \geq 3$, and define $\mathcal{G}_{2N}^{\sigma}$ via eq.* (3.3.2)*. Suppose that the initial data (u_0, η_0) satisfy $\mathcal{E}_{2N}^{\sigma}(0) + \mathcal{F}_{2N}(0) < \infty$ as well as compatibility conditions of theorem 3.2.2. There exist universal constants $\gamma_0, \kappa_0 \in (0,1)$ and $0 < c < g$ such that if $\mathcal{E}_{2N}^{\sigma}(0) + \mathcal{F}_{2N}(0) \leq \kappa_0, 0 \leq \sum_{\ell=2}^{2N+2} A\omega^{\ell} \leq \gamma_0$, $0 \leq \sigma \leq 1$, and $g - A\omega^2 > c$, then there exists a unique triple (u, p, η) that solves the (2.3.10) on the temporal interval $(0, \infty)$, achieves the initial data, and obeys the estimate*

$$\mathcal{G}_{2N}^{\sigma}(\infty) \lesssim \mathcal{E}_{2N}^{\sigma}(0) + \mathcal{F}_{2N}(0). \tag{3.3.3}$$

In particular, the bound in theorem 3.3.2 establishes the decay estimate

$$\mathcal{E}_{N+2}^{\sigma}(t) \lesssim \frac{\mathcal{E}_{2N}^{\sigma}(0) + \mathcal{F}_{2N}(0)}{(1+t)^{4N-8}}. \tag{3.3.4}$$

This is an algebraic decay rate, slower than the exponential rate proved in theorem 3.3.1 with a fixed $\sigma > 0$. Two remarks about this are in order. First, by chosing N larger, we arrive at a faster rate of decary. In fact, by taking N to be arbitrarily large we can achieve arbitrarily fast algebraic decay rates, which is what is known as "almost exponential decay". Of course, the trade-off in the theorem is that faster decay requires smaller data in higher regularity classes. The second point is that when $0 < \sigma \leq 1$ in the theorem, it is still possible to prove that $\mathcal{E}_{2n}^{\sigma}$ decays exponentially by modifying the arguments used later in theorem 8.1.1. We neglect to state this properly here because we only care about the vanishing surface tension limit, and in this case we cannot get uniform control of the exponential decay parameter $\lambda(\sigma)$ from theorem 3.3.1.

Theorem 3.3.2 also guarantees enough regularity to switch back to Eulerian coordinates. Consequently, the theorem tells us that the steady oscillating solution in eq. (2.2.1) remains asymptotically stable without surface tension, but that the rate of decay to equilibrium is slower.

Our third result establishes the vanishing surface tension limit for the problem eq. (2.3.10).

Theorem 3.3.3. *Let Ω be given by eq.* (2.3.1)*, let $N \geq 3$, and consider a decreasing sequence $\{\sigma_m\}_{m=0}^{\infty} \subset (0,1)$ such that $\sigma_m \to 0$ as $m \to \infty$. Let $\kappa_0, \gamma_0 \in (0,1)$ be as in theorem 3.3.2, and assume that $0 \leq \sum_{\ell=2}^{2N+2} A\omega^{\ell} \leq \gamma_0$. Suppose that for each $m \in \mathbb{N}$ we have initial data $(u_0^{(m)}, \eta_0^{(m)})$ satisfying $\mathcal{E}_{2N}^{\sigma_m}(0) + \mathcal{F}_{2N}(0) < \kappa_0$ as well as the comptability conditions of theorem 3.2.2. Let $(u^{(m)}, p^{(m)}, \eta^{(m)})$ be the global solutions to eq.* (2.3.10) *associated to the data given by theorem 3.3.2. Further assume that*

$$u_0^{(m)} \to u_0 \text{ in } H^{4N}(\Omega),\ \eta_0^{(m)} \to \eta_0 \text{ in } H^{4N+1/2}(\Sigma), \text{ and } \sqrt{\sigma_m} \nabla \eta_0^{(m)} \to 0 \text{ in } H^{4N}(\Sigma) \tag{3.3.5}$$

as $m \to \infty$.

Then the following hold.

1. *The pair (u_0, η_0) satisfy the compatibility conditions of theorem 3.2.2 with $\sigma = 0$.*

2. *As $m \to \infty$, the triple $(u^{(m)}, p^{(m)}, \eta^{(m)})$ converges to (u, p, η), where the latter triple is the unique solution to eq. (2.3.10) with $\sigma = 0$ and initial data (u_0, η_0). The convergence occurs in any space into which the space of triples (u, p, η) obeying $\mathcal{G}^0_{2N}(\infty) < \infty$ compactly embeds.*

3.4 Discussion and plan of paper

The flavor of this paper is very similar to those in past literature such as [GT13, Tic18]. As in these papers, the main focus of this paper is to establish a priori estimates for solutions to the PDE eq. (2.3.10), which allows us to conclude theorems 3.3.1 to 3.3.2 by standard arguments coupling these estimates with local existence results. The scheme of a priori estimates developed in this paper are a variant of the nonlinear energy method employed in [GT13, Tic18] that is designed to carefully track the dependence on the magnitude parameter A and frequency parameter ω in order to optimize the parameter regime in which we obtain the desired existence and stability theorem. More specifically, we show that the desired theorem in the surface tension case can be obtained in the parameter regime of

$$A\omega^2 + A\omega^3 \lesssim 1. \tag{3.4.1}$$

Thus, the Faraday oscillation system can be stable for arbitrarily large A or ω, so long as the other parameter is sufficiently small. In the vanishing surface tension case, we obtain a similar result, although the trade-off is that more stringent constraints on A and ω are necessary.

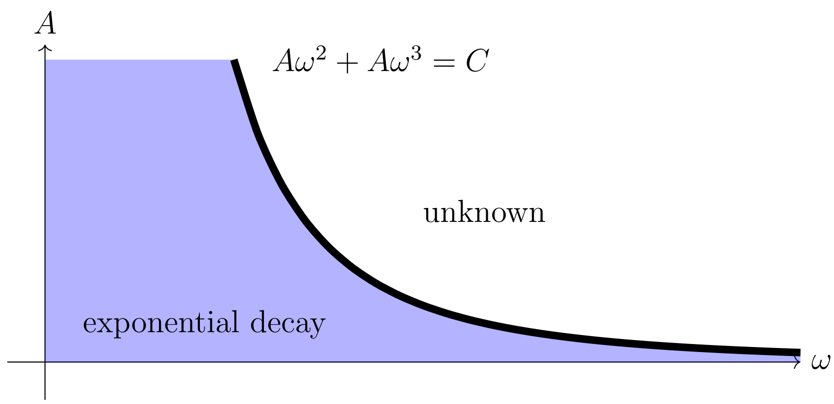

Figure 3.1: A graph of the parameter regime in which we have asymptotic stability.

We now outline the main steps of the scheme.

Horizontal energy estimates: The main workhorse in our analysis is the energy-dissipation equation eq. (2.2.2) and its linearized counterparts. The natural physical energy and dissipation in eq. (2.2.2) do not provide nearly enough control to close a scheme of a priori estimates, so we are forced to seek control of higher-order derivatives by applying derivatives to the PDE eq. (2.3.10) and again appealing to enegy-dissipation relations. For this to work, the differential operators need to be compatible with the boundary conditions of eq. (2.3.10), and this restricts us to "horizontal" derivatives of the form ∂^α for $\alpha \in \mathbb{N}^{1+2}$.

It turns out to be convenient to do these estimates in two forms, one for the equations written in "geometric" as in eq. (4.1.1), and the second written in "flattened" form as in eq. (4.2.1). The geometric form is better suited for analysis of the time derivative version of the energy-dissipation, as it circumvents the problem of estimating $\partial_t p$, which is not controlled by our scheme of estimates. The flattened form works well for other derivatives and is convenient to use due to its compatibility with related elliptic estimates and due to the somewhat simpler constant coefficient form. We develop these forms of the energy-dissipation relations in chapter 4.

Summing the various energy-dissipation relations over a range of derivatives provides us with an equation roughly of the form

$$\frac{d}{dt}\overline{\mathcal{E}_n^\sigma} + \overline{\mathcal{D}_n} = \left(\sum_{\ell=2}^{N} A\omega^\ell\right)\mathcal{J}_n + \mathcal{I}_n, \tag{3.4.2}$$

where $\mathcal{I}_n$ is a cubic (or higher) interaction energy and $\mathcal{J}_n$ is a quadratic term generated by the oscillation background. The more precise form of eq. (3.4.2) and its proof can be found in the first part of chapter 6.

It is important to note that the interaction term $\mathcal{I}_n$ involves more differential operators than are controlled by either $\mathcal{E}_n^\sigma$ or $\mathcal{D}_n^\sigma$, so a nonlinear energy method based solely on the horizontal terms is impossible. We are thus compelled to appeal to auxiliary estimates in order to gain control of more terms.

Energy and dissipation enhancement: The next step in the nonlinear energy method is to employ various auxiliary estimates in order to gain control of more quantities in terms of those already controlled by $\overline{\mathcal{E}_n^\sigma}$ and $\overline{\mathcal{D}_n}$. In other words, we seek to prove (again, roughly) that we have the comparison estimates

$$\mathcal{E}_n^\sigma \lesssim \overline{\mathcal{E}_n^\sigma} \leq \mathcal{E}_n^\sigma \qquad \mathcal{D}_n^\sigma \lesssim \overline{\mathcal{D}_n} \leq \mathcal{D}_n^\sigma. \tag{3.4.3}$$

The main mechanisms for proving eq. (3.4.3) are elliptic regularity for the Stokes problem and elliptic regularity for the capillary problem, both of which are recorded in appendix B. The proof of eq. (3.4.3), up to error terms, is recorded in the latter part of chapter 6.

Nonlinear estimates: In eq. (3.4.3) we record estimates of the various nonlinear terms that appear in the energy-dissipation, elliptic, and auxiliary estimates employed in the analysis. A good portion of the nonlinearities can be handled in the usual way with a combination of Sobolev space product estimates, embeddings, and trace estimates. However, a few of the nonlinearities present key challenges and must be treated delicately in order to arrive at a useful estimate.

The nonlinear energy method requires more than just control of the nonlinearities: it requires structured control. The rough idea here is that the nonlinear terms must be able to be absorbed by the dissipation functional in a small energy context. For example, it is not enough to bound the $\mathcal{I}_n$ term mentioned above via $|\mathcal{I}_n| \lesssim (\mathcal{D}_n^\sigma)^r$ for some $r > 1$. We must instead have an estimate that is structured in a way compatible with absorption, i.e. one of the form

$$|\mathcal{I}_n| \lesssim (\mathcal{E}_n^\sigma)^r \mathcal{D}_n^\sigma \tag{3.4.4}$$

for some $r > 0$. With such an estimate in hand we can work in a small-energy context, i.e. in the context of $\mathcal{E}_n^\sigma \ll 1$, in order to view $|\mathcal{I}_n|$ as a small multiple of the dissipation.

A priori estimates with zero or vanishing surface tension: When $\sigma = 0$ and in the vanishing surface tension analysis we cannot exploit the regularity gains afforded by the elliptic capillary problem. This creates two serious problems. The first is that without this control the dissipation fails to be coercive over the energy precisely due to a half-derivative gap in the estimate for η. This means we can no longer expect exponential decay of solutions. The second and more severe problem is that the nonlinear estimates require control of $2n+1/2$ derivatives of η, where as the energy only controls $2n$ derivatives of η. This disparity is potentially disastrous even for the local existence theory, as it suggests derivative loss. Fortunately, the kinematic transport equation for η provides an alternate way of estimating these derivatives and shows that they are finite. Unfortunately, the best estimates associated to the transport equation give rise to bounds that grow linearly in time, and this poses serious problems for a nonlinear energy method in which the nonliearity is supposed to be small in some sense.

We get around these problems by employing the two-tier nonlinear energy method developed in [GT13]. The idea is to get $N \geq 3$ and consider together the high-order energy and dissipation $\mathcal{E}^0_{2N}$ and $\mathcal{D}^0_{2N}$ along with the low-order energy and dissipation $\mathcal{E}^0_{N+2}$ and $\mathcal{D}^0_{N+2}$. We also use the functional $\mathcal{F}_{2N}$ defined by eq. (3.1.6) to track the highest derivatives of η.

We control $\mathcal{F}_{2N}$ with a transport estimate in the first part of Section 6, but the estimate allows for $\mathcal{F}_{2N}$ to grow linearly in time. To compensate for this in the nonlinear estimate of Section 4 we show that $\mathcal{F}_{2N}$ only appears in products with the very low regularity functional $\mathcal{K}$ defined by eq. (3.1.7). We have a trivial estimate $\mathcal{K} \lesssim \mathcal{E}^0_{N+2}$, and so if we know a priori decays algebraically at a fast enough rate, then the product $\mathcal{F}_{2N}\mathcal{K}$ can be controlled uniformly in time. Then, under the assumptions that $\mathcal{G}^0_{2N} \ll 1$ we prove that (again, roughly)

$$\sup_{0 \leq t \leq T} \mathcal{E}^0_{2N}(t) + \int_0^T \mathcal{D}^0_{2N}(t)\, dt \lesssim \mathcal{E}^0_{2N}(0). \tag{3.4.5}$$

This means that the decay of the low-tier energy allows us to close the high-tier bounds.

Next we use the high-tier bounds to show that the low-tier energy decays algebraically, with bounds in terms of the data. The key point here is that $\mathcal{D}^0_{N+2}$ is not coercive over $\mathcal{E}^0_{N+2}$, but it is possible to interpolate with the high-energy bound:

$$\mathcal{E}^0_{N+2} \lesssim (\mathcal{D}^0_{N+2})^{\theta} (\mathcal{E}^0_{2N})^{1-\theta} \tag{3.4.6}$$

for some $\theta = \theta(N) \simeq 1$. This then allows us to prove a bound of the form

$$\frac{d}{dt}\bar{\mathcal{E}}_{N+2} + C(\bar{\mathcal{E}}_{N+2})^{1+1/r} \leq 0 \tag{3.4.7}$$

for some $r = r(N) > 0$, and from this we can deduce the decay estimate

$$\mathcal{E}_{N+2}(t) \lesssim \mathcal{E}_{2N}(0)(1+t)^{-r}. \tag{3.4.8}$$

This means that the boundedness of the high-tier energy allows us to close the low-tier decay bounds.

The second part of section 7 contains the full details of the two-tier method and establishes the global well-posedness and algebraic decay of solutions. An interesting feature of the two-tier analysis is that the existence of global solutions is predicated on their decay.

Vanishing surface tension limit: Throughout the paper we take great care to isolate the behavior of constants with respect to σ. This is done in order to allow us to send $\sigma \to 0$. We carry out this analysis in the final part of section 7.

Chapter 4

Evolution of the energy and dissipation

In this section we record the energy-dissipation evolution equations for two linearized versions of the problem eq. (2.3.10): the geometric form and the flattened form. We also record the forms of the nonlinear forcing terms that appear in the analysis of eq. (2.3.10).

4.1 Geometric form

Let $\Phi, \mathcal{A}, \mathcal{N}, J$, etc. be given in terms of η as before. We give the geometric linearization of eq. (2.3.10) for (v, q, ζ):

$$\begin{cases} \partial_t v - \partial_t \hat{\eta} \tilde{b} K \partial_3 q + u \cdot \nabla_{\mathcal{A}} \zeta + \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(v, q) = \Psi^1 & \text{in } \Omega \\ \operatorname{div}_{\mathcal{A}} v = \Psi^2 & \text{in } \Omega \\ \partial_t \zeta - v \cdot \mathcal{N} = \Psi^3 & \text{on } \Sigma \\ S_{\mathcal{A}}(v, q) \mathcal{N} = (-\sigma \Delta \zeta + g\zeta + \Psi^5) \mathcal{N} + \Psi^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b \end{cases} \tag{4.1.1}$$

4.1.1 Energy-dissipation

The next result records the energy-dissipation equation associated to the solutions of eq. (4.1.1).

Proposition 4.1.1 (Geometric energy-dissipation)**.** *Let η and u be given and satisfy*

$$\begin{cases} \operatorname{div}_{\mathcal{A}} u = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \end{cases}. \tag{4.1.2}$$

Suppose that (v, q, ζ) solve eq. (4.1.1), where $\Phi, \mathcal{A}, J$, etc. are determined by η as before. Then,

$$\begin{aligned} &\frac{d}{dt}\left[\int_\Omega \frac{|v|^2 J}{2} + \int_\Sigma \frac{\sigma |\nabla \eta|^2}{2} + \frac{g|\eta|^2}{2}\right] + \int_\Omega \mu \frac{|\mathbb{D}_{\mathcal{A}} v|^2 J}{2} \\ &\quad = \int_\Omega J\left(v \cdot \Psi^1 + q\Psi^2\right) + \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta\right)\Psi^3 - \Psi^4 \cdot (\partial_t^n u) - \Psi^5 v \cdot \mathcal{N}. \end{aligned} \tag{4.1.3}$$

Proof. We take the dot product of the first equation in eq. (4.1.1) with v, multiply by J, and integrate over Ω to see that

$$I + II = \int_\Omega \Psi^1 \cdot vJ \tag{4.1.4}$$

for

$$\begin{aligned} I &= \int_\Omega \partial_t v \cdot vJ - \partial_t \hat{\eta}\tilde{b}\partial_3 v \cdot v + (u \cdot \nabla_{\mathcal{A}} v) \cdot vJ \\ II &= \int_\Omega \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(v,q) \cdot vJ \end{aligned} . \tag{4.1.5}$$

In order to integrate these terms by parts, we will utilize the geometric identity $\partial_k(J\mathcal{A}_{ik}) = 0$ (details in section appendix A.3) for each i.

Computing I To handle the term I, we first compute

$$I = \partial_t \int_\Omega \frac{|v|^2 J}{2} + \int_\Omega -\frac{|v|^2 \partial_t J}{2} - \partial_t \hat{\eta}\tilde{b}\partial_3 \frac{|v|^2}{2} + u_j \partial_k \left(J\mathcal{A}_{jk} \frac{|v|^2}{2} \right) =: I_1 + I_2. \tag{4.1.6}$$

Since $\tilde{b} = (1 + x_3/b)$, an integration by parts, an application of the boundary condition $v = 0$ on Σ_b reveals that

$$\begin{aligned} I_2 &= \int_\Omega -\frac{|v|^2 \partial_t J}{2} - \partial_t \hat{\eta}\tilde{b}\partial_3 \frac{|v|^2}{2} + u_j \partial_k \left(J\mathcal{A}_{jk} \frac{|v|^2}{2} \right) \\ &= \int_\Omega -\frac{|v|^2 \partial_t J}{2} + \frac{|v|^2}{2}\partial_3 \left(\partial_t \hat{\eta}\tilde{b} \right) + u_j \partial_k \left(J\mathcal{A}_{jk} \frac{|v|^2}{2} \right) + \int_\Sigma -\frac{|v|^2}{2}\partial_t \hat{\eta}\tilde{b} \\ &= \int_\Omega -\frac{|v|^2}{2} \left(\frac{\partial \hat{\eta}}{b} + \partial_3 \partial_t \hat{\eta}\tilde{b} \right) + \frac{|v|^2}{2} \left(\partial_3 \partial_t \hat{\eta}\tilde{b} + \frac{\partial_t \hat{\eta}}{b} \right) - (\partial_k u_j) \left(J\mathcal{A}_{jk} \frac{|v|^2}{2} \right) \\ &\qquad + \int_\Sigma -\frac{|v|^2}{2}\partial_t \hat{\eta} + u_j J\mathcal{A}_{jk} \frac{|v|^2}{2}(e_3 \cdot e_k) \\ &= \int_\Omega -J\frac{|v|^2}{2} \operatorname{div}_{\mathcal{A}} u + \int_\Sigma -\frac{|v|^2}{2}\partial_t \hat{\eta} + u_j J\mathcal{A}_{jk} \frac{|v|^2}{2}(e_3 \cdot e_k). \end{aligned} \tag{4.1.7}$$

Now note that $J\mathcal{A}_{jk}(e_3 \cdot e_k) = \mathcal{N}_j$ on Σ and also we have that u and η satisfy eq. (4.1.2), so the above becomes

$$I_2 = \int_\Omega -J\frac{|v|^2}{2} \operatorname{div}_{\mathcal{A}} u + \int_\Sigma \frac{|v|^2}{2} \left(-\partial_t \eta + u \cdot \mathcal{N} \right) = 0 \tag{4.1.8}$$

and hence

$$I = I_1 + I_2 = \partial_t \int_\Omega \frac{|v|^2 J}{2} \tag{4.1.9}$$

so I is purely just the transport of the quantity $|v|^2 J$ along the flow u.

Computing II We begin our analysis of the term II with a similar integration by parts, which reveals that

$$\begin{aligned}
II &= \int_\Omega \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(v,q)\cdot vJ = \int_\Omega \mathcal{A}_{jk}\partial_k\left(S_{\mathcal{A}}(v,q)\right)_{ij} v_i J = \int_\Omega v_i J\mathcal{A}_{jk}\partial_k\left(S_{\mathcal{A}}(v,q)\right)_{ij} \\
&= \int_\Omega -\partial_k\left(v_i J\mathcal{A}_{jk}\right)\left(S_{\mathcal{A}}(v,q)\right)_{ij} + \int_\Sigma v_i J\mathcal{A}_{jk}\left(S_{\mathcal{A}}(v,q)\right)_{ij}(e_3\cdot e_k) \\
&= \int_\Omega -\left[\partial_k\left(v_i J\mathcal{A}_{jk}\right) - v_i\partial_k\left(J\mathcal{A}_{jk}\right)\right]\left(S_{\mathcal{A}}(v,q)\right)_{ij} + \int_\Sigma v_i J\mathcal{A}_{j3}\left(S_{\mathcal{A}}(v,q)\right)_{ij} \\
&= \int_\Omega -J\mathcal{A}_{jk}\partial_k v_i\left(S_{\mathcal{A}}(v,q)\right)_{ij} + \int_\Sigma v_i\left(S_{\mathcal{A}}(v,q)\right)_{ij}\mathcal{A}_{j3}J \\
&= \int_\Omega -J\left(\nabla_{\mathcal{A}}v\right)_{ij}\left(S_{\mathcal{A}}(v,q)\right)_{ij} + \int_\Sigma v_i\left(S_{\mathcal{A}}(v,q)\right)_{ij}\mathcal{N}_j \\
&= \int_\Omega -JS_{\mathcal{A}}(v,q) : \nabla_{\mathcal{A}}v + \int_\Sigma S_{\mathcal{A}}(v,q)\mathcal{N}\cdot v \\
&= \int_\Omega -J\left(q\operatorname{div}_{\mathcal{A}} v - \frac{\mu|\mathbb{D}_{\mathcal{A}}v|^2}{2}\right) + \int_\Sigma S_{\mathcal{A}}(v,q)\mathcal{N}\cdot v \\
&= \int_\Omega -J\left(q\Psi^2 - \frac{\mu|\mathbb{D}_{\mathcal{A}}v|^2}{2}\right) + \int_\Sigma S_{\mathcal{A}}(v,q)\mathcal{N}\cdot v.
\end{aligned} \tag{4.1.10}$$

Now using the third and fourth equations in eq. (4.1.1), we rewrite the integral on Σ as

$$\begin{aligned}
\int_\Sigma S_{\mathcal{A}}(v,q)\mathcal{N}\cdot v &= \int_\Sigma \left[\left(-\sigma\Delta\zeta + g\zeta + \Psi^5\right)\mathcal{N} + \Psi^4\right]\cdot v \\
&= \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta + \Psi^5\right)\mathcal{N}\cdot v + \int_\Sigma \Psi^4\cdot v \\
&= \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta\right)\mathcal{N}\cdot v + \int_\Sigma \Psi^4\cdot v + \Psi^5 v\cdot\mathcal{N} \\
&= \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta\right)\left(\partial_t\zeta - \Psi^3\right) + \int_\Sigma \Psi^4\cdot v + \Psi^5 v\cdot\mathcal{N} \\
&= \partial_t\left[\int_\Sigma \frac{\sigma|\nabla\zeta|^2}{2} + \frac{g|\zeta|^2}{2}\right] - \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta\right)\Psi^3 + \int_\Sigma \Psi^4\cdot v + \Psi^5 v\cdot\mathcal{N}
\end{aligned} \tag{4.1.11}$$

so on sum, we have

$$\begin{aligned}
II = \int_\Omega -J\left(q\Psi^2 - \frac{\mu|\mathbb{D}_{\mathcal{A}}v|^2}{2}\right) + \frac{d}{dt}\left[\int_\Sigma \frac{\sigma|\nabla\zeta|^2}{2} + \frac{g|\zeta|^2}{2}\right] \\
- \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta\right)\Psi^3 + \int_\Sigma \Psi^4\cdot v + \Psi^5 v\cdot\mathcal{N}.
\end{aligned} \tag{4.1.12}$$

Now to see that equation eq. (4.1.3) holds, just plug eq. (4.1.9) and eq. (4.1.12) into eq. (4.1.4) and rearrange. □

4.1.2 Forcing terms

We now record the form of the forcing terms that will appear in our analysis. Recall that this geometric form of the linearization is responsible for the highest order time derivatives ∂_t^n, so we build this into the notation by writing $F^{j,n}$ for the jth forcing term generated by applying ∂_t^n to eq. (2.3.10).

Applying ∂_t^n to the ith component of the first equation results in

$$\begin{aligned} \partial_t(\partial_t^n u_i) + \sum_{0\le \ell \le n} C_{\ell n}\left[-\partial_t^\ell\left(\partial_t \hat{\eta}\tilde{b}K\right)\partial_t^{n-\ell}\left(\partial_3 u_i\right) + \partial_t^\ell\left(u_j \mathcal{A}_{jk}\right)\partial_t^{n-\ell}\left(\partial_k u_i\right)\right] \\ + \sum_{0\le \ell \le n} C_{\ell n}\left[\partial_t^\ell \mathcal{A}_{jk}\partial_t^{n-\ell}\partial_k\left(S_{\mathcal{A}}(u,p)\right)_{ij}\right] = 0. \end{aligned} \tag{4.1.13}$$

Then in the above, the first term as well as the terms in the summation corresponding to $\ell = 0$ gives the left hand side of eq. (4.1.1), except for the last sum for which we have an extra term

$$\begin{aligned} \mathcal{A}_{jk}\partial_t^n\partial_k\left(S_{\mathcal{A}}(u,p)_{ij}\right) - \mathcal{A}_{jk}\partial_k\left(S_{\mathcal{A}}(\partial^n u, \partial_t^n p)_{ij}\right) &= \mathcal{A}_{jk}\left(-\sum_{0\le \ell \le n} C_{\ell n}\mu \mathbb{D}_{\partial_t^\ell \mathcal{A}}\partial_t^{n-\ell} u + \mu \mathbb{D}_{\mathcal{A}}\partial_t^n u\right) \\ &= -\mathcal{A}_{jk}\sum_{0< \ell \le n} C_{\ell n}\mu \mathbb{D}_{\partial_t^\ell \mathcal{A}}\partial_t^{n-\ell} u. \end{aligned} \tag{4.1.14}$$

Thus,

$$\begin{aligned} F_i^{1,n} = \sum_{0< \ell \le n} C_{\ell n}\left[\partial_t^\ell\left(\partial_t \hat{\eta}\tilde{b}K\right)\partial_t^{n-\ell}\left(\partial_3 u_i\right) - \partial_t^\ell\left(u_j \mathcal{A}_{jk}\right)\partial_t^{n-\ell}\left(\partial_k u_i\right)\right] \\ + \sum_{0< \ell \le n} C_{\ell n}\left[-\partial_t^\ell \mathcal{A}_{jk}\partial_t^{n-\ell}\partial_k\left(S_{\mathcal{A}}(u,p)\right)_{ij} + \mathcal{A}_{jk}\mu \mathbb{D}_{\partial^\ell \mathcal{A}}\partial_t^{n-\ell} u\right] = 0. \end{aligned} \tag{4.1.15}$$

Differentiating the second equation gives

$$\sum_{0\le \ell \le n} C_{\ell n}\partial_t^\ell \mathcal{A}_{jk}\partial_t^{n-\ell}\partial_k u_j = 0 \tag{4.1.16}$$

so taking all but the $\ell = 0$ terms gives

$$F^{2,n} = \sum_{0< \ell \le n} C_{\ell n}\partial_t^\ell \mathcal{A}_{jk}\partial_t^{n-\ell}\partial_k u_j. \tag{4.1.17}$$

Differentiating the third equation gives

$$\partial_t(\partial_t^n \eta) - \sum_{0\le \ell \le n} C_{\ell n}\partial_t^\ell u \cdot \partial_t^{n-\ell}\mathcal{N} = 0 \tag{4.1.18}$$

so

$$F^{3,n} = \sum_{0< \ell \le n} C_{\ell n}\partial_t^\ell u \cdot \partial_t^{n-\ell}\mathcal{N}. \tag{4.1.19}$$

Finally, differentiating the ith component of the fourth equation gives

$$\begin{aligned} \sum_{0\le \ell \le n} C_{\ell n}\partial_t^{n-\ell}\left(S_{\mathcal{A}}(u,p)\right)_{ij}\partial_t^\ell \mathcal{N}_j &= \sum_{0\le \ell \le n} C_{\ell n}\partial_t^{n-\ell}\left(-\sigma \mathfrak{H}(\eta) + \left(g + A\omega^2 f''(\omega t)\right)\eta\right)\partial_t^\ell \mathcal{N}_i \\ &= \sum_{0\le \ell \le n} C_{\ell n}\left(-\sigma \partial_t^{n-\ell}\mathfrak{H}(\eta) + g\partial_t^{n-\ell}\eta + \partial_t^{n-\ell}\left(A\omega^2 f''(\omega t)\eta\right)\right)\partial_t^\ell \mathcal{N}_i \end{aligned} \tag{4.1.20}$$

so taking away the $\ell = 0$ terms and handling $\partial_t^n \left(S_{\mathcal{A}}(u,p)\right)_{ij}$ as before as well as handling the $\Delta\eta$ term, we get

$$\begin{aligned} F_i^{4,n} &= \sum_{0<\ell\leq n} C_{\ell n} \left[-\partial_t^{n-\ell} \left(S_{\mathcal{A}}(u,p)\right)_{ij} \partial_t^{\ell} \mathcal{N}_j + \left(\mu \mathbb{D}_{\partial^\ell \mathcal{A}} \partial_t^{n-\ell} u\right)_{ij} \mathcal{N}_j \right] \\ &\quad + \sum_{0<\ell\leq n} C_{\ell n} \left(-\sigma \partial_t^{n-\ell} \mathfrak{H}(\eta) + g \partial_t^{n-\ell} \eta + \partial_t^{n-\ell} \left(A\omega^2 f''(\omega t)\eta \right) \right) \partial_t^{n-\ell} \mathcal{N}_i + \left(-\sigma \partial_t^n \left(\mathfrak{H}(\eta) - \Delta\eta \right) \right) \mathcal{N}_i \end{aligned} \tag{4.1.21}$$

and

$$F^{5,n} = \partial_t^n \left(A\omega^2 f''(\omega t)\eta \right). \tag{4.1.22}$$

4.2 Flattened form

It will also be useful for us to have a linearized version of eq. (2.3.10) with constant coefficients. This version is as follows (details in section appendix A.4):

$$\begin{cases} \partial_t v + \operatorname{div} S(v,q) = \Theta^1 & \text{in } \Omega \\ \operatorname{div} v = \Theta^2 & \text{in } \Omega \\ \partial_t \zeta = v_3 + \Theta^3 & \text{on } \Sigma \\ S(v,q)e_3 = \left(-\sigma\Delta\zeta + g\zeta + \Theta^5\right) e_3 + \Theta^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b \end{cases}. \tag{4.2.1}$$

4.2.1 Energy-dissipation

Proposition 4.2.1 (Flattened energy-dissipation)**.** *Suppose* (v, q, ζ) *solve eq.* (4.2.1)*. Then*

$$\begin{aligned} &\frac{d}{dt}\left[\int_\Omega \frac{|v|^2}{2} + \int_\Sigma \frac{\sigma|\nabla\zeta|^2}{2} + \frac{g|\zeta|^2}{2} \right] + \frac{\mu}{2} \int_\Omega |\mathbb{D}v|^2 \\ &= \int_\Omega v \cdot \Theta^1 + q\Theta^2 + \int_\Sigma \left(-\sigma\Delta\zeta + g\zeta\right) \Theta^3 - \Theta^4 \cdot v - \Theta^5 v_3. \end{aligned} \tag{4.2.2}$$

Proof. We dot the first equation of eq. (4.2.1) with v and integrate over Ω to see that

$$I + II = \int_\Omega v \cdot \Theta^1 \tag{4.2.3}$$

where

$$I := \int_\Omega v \partial_t v, \qquad II := \int_\Omega v \cdot \nabla q - \mu v \cdot \Delta v. \tag{4.2.4}$$

We trivially have

$$I = \partial_t \int_\Omega \frac{|v|^2}{2}. \tag{4.2.5}$$

To deal with II, recall that

$$\nabla q - \mu\Delta v = \operatorname{div}(qI - \mu\mathbb{D}v), \tag{4.2.6}$$

and so

$$\begin{aligned} \int_\Omega v \cdot (\nabla q - \mu\Delta v) &= \int_\Omega v \cdot \operatorname{div}(qI - \mu\mathbb{D}v) \\ &= \int_\Omega -(qI - \mu\mathbb{D}v) : \nabla v + \int_\Sigma (qI - \mu\mathbb{D}v)e_3 \cdot v := II_1 + II_2. \end{aligned} \tag{4.2.7}$$

A simple computation gives

$$II_1 = \int_\Omega \mu \mathbb{D} v : \nabla v - (qI) : \nabla v = \frac{\mu}{2} \int_\Omega |\mathbb{D} v|^2 - \int_\Omega \operatorname{div}(qv) = \frac{\mu}{2} \int_\Omega |\mathbb{D} v|^2 - q\Theta^2. \tag{4.2.8}$$

Now

$$\begin{aligned} II_2 &= \int_\Sigma v \cdot \left[(-\sigma \Delta \zeta + g\zeta + \Theta^5) e_3 + \Theta^4 \right] \\ &= \int_\Sigma \left(-\sigma \Delta \zeta + g\zeta + \Theta^5 \right) v_3 + \Theta^4 \cdot v \\ &= \int_\Sigma (-\sigma \Delta \zeta + g\zeta) (\partial_t \zeta - \Theta^3) + \Theta^4 \cdot v + \Theta^5 v_3 \\ &= \partial_t \left[\int_\Sigma \frac{\sigma |\nabla \zeta|^2}{2} + \frac{g|\zeta|^2}{2} \right] - (-\sigma \Delta \zeta + g\zeta)\Theta^3 + \Theta^4 \cdot v + \Theta^5 v_3. \end{aligned} \tag{4.2.9}$$

Thus, in sum, we have

$$II = \partial_t \left[\int_\Sigma \frac{\sigma |\nabla \zeta|^2}{2} + \frac{g|\zeta|^2}{2} \right] + \frac{\mu}{2} \int_\Omega |\mathbb{D} v|^2 - \int_\Omega q\Theta^2 - (-\sigma \Delta \zeta + g\zeta)\Theta^3 + \Theta^4 \cdot v + \Theta^5 v_3. \tag{4.2.10}$$

The result follows by addition and regrouping. □

4.2.2 Forcing terms

The forcing terms come from rearranging the equation to get the terms we want – we then designate everything else as forcing terms. Note that we will take derivatives of the full nonlinear equations in eq. (2.3.10), but to get the corresponding forcing terms, we may just take derivatives of the forcing terms here since we constructed our linearization to have constant coefficients. To get the first forcing term, remark that the first equation in eq. (2.3.10) can be rewritten as

$$\begin{aligned} \partial_t u + \operatorname{div} S(u,p) &= \partial_t \hat{\eta} \tilde{b} K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u + (\operatorname{div} S(u,p) - \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(u,p)) \\ &= \partial_t \hat{\eta} \tilde{b} K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u - \operatorname{div} \mathbb{D}_{I-\mathcal{A}} u - \operatorname{div}_{\mathcal{A}-I} (pI - \mathbb{D}_{\mathcal{A}} u), \end{aligned} \tag{4.2.11}$$

and so

$$G^1 = \partial_t \hat{\eta} \tilde{b} K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u - \operatorname{div} \mathbb{D}_{I-\mathcal{A}} u - \operatorname{div}_{\mathcal{A}-I} (pI - \mathbb{D}_{\mathcal{A}} u). \tag{4.2.12}$$

The second term is

$$G^2 = \operatorname{div}_{I-\mathcal{A}} u; \tag{4.2.13}$$

this is a result of simply adding and subtracting the two different types of divergence. To handle the third equation, rewrite as

$$\partial_t \eta = u \cdot e_3 + u \cdot (\mathcal{N} - e_3), \tag{4.2.14}$$

and so

$$G^3 = u \cdot (\mathcal{N} - e_3). \tag{4.2.15}$$

Finally, we similarly write the fourth nonlinear term as

$$\begin{aligned} G^4 = (pI - \mu \mathbb{D} u)(e_3 - \mathcal{N}) + (\mu \mathbb{D}_{\mathcal{A}-I} u)\mathcal{N} + \left(g\eta + A\omega^2 f''(\omega t)\eta \right)(e_3 - \mathcal{N}) \\ - (-\sigma \mathfrak{H}(\eta))(e_3 - \mathcal{N}) - (-\sigma (\Delta \eta - \mathfrak{H}(\eta))) e_3 \end{aligned} \tag{4.2.16}$$

and the fifth linear error term as

$$G^5 = A\omega^2 f''(\omega t)\eta.$$

Chapter 5

Estimates of the nonlinearities and other error terms

In this section we develop the estimates of the nonlinearities as well as other error terms needed to close our scheme of a priori estimates.

5.1 L^∞ estimates

The next result establishes some key L^∞ bounds that will be used repeatedly throughout the paper.

Proposition 5.1.1. *There exists a universal constant $\delta \in (0,1)$ such that if $\|\eta\|_1^2 \le \delta$, then the following bounds hold.*

1. *We have that*
$$\|J-1\|_{L^\infty}^2 + \|\mathcal{N}-e_3\|_{L^\infty}^2 + \|A\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \le \frac{1}{2} \tag{5.1.1}$$
and
$$\|K\|_{L^\infty}^2 + \|\mathcal{A}\|_{L^\infty}^2 \lesssim 1. \tag{5.1.2}$$

2. *The mapping given by eq.* (2.3.3) *is a diffeomorphism from Ω to $\Omega(t)$.*

3. *For all $v \in H^1(\Omega)$ such that $v = 0$ on Σ_b we have the estimate*
$$\int_\Omega |\mathbb{D}v|^2 \le \int_\Omega J|\mathbb{D}_{\mathcal{A}}v|^2 + C\left(\|\mathcal{A}-I\|_{L^\infty} + \|J-1\|_{L^\infty}\right)\int_\Omega |\mathbb{D}v|^2 \tag{5.1.3}$$
for a universal constant $C > 0$.

Proof. Recall that
$$J - 1 = \frac{\hat{\eta}}{b} + \partial_3\hat{\eta}\tilde{b}, \qquad \mathcal{N} - e_3 = (-\partial_1\eta, -\partial_2\eta, 0), \qquad A = \partial_1\hat{\eta}\tilde{b}, \qquad B = \partial_2\hat{\eta}\tilde{b}. \tag{5.1.4}$$

Thus, the left hand side of eq. (5.1.1) can be bounded above Sobolev embedding $H^3(\Omega) \hookrightarrow C^1(\Omega)$ by $\|\hat{\eta}\|_3$. This is in turn bounded by $\|\eta\|_{5/2}$ by lemma C.2.2. Then eq. (5.1.2) holds by the definitions of K and $\mathcal{A}$ and eq. (5.1.1). To see the second item, simply note that we have a diffeomorphism if and only if the Jacobian of Φ, J, is nonzero. This can be accomplished by taking $\|\hat{\eta}\|_{L^\infty} < b$. Note that for the $\mathcal{N} - e_3$ term, we bound the L^∞ of η on Σ by L^∞ of $\hat{\eta}$ on Ω in L^∞ by sufficient regularity. For the third item, first write

$$\begin{aligned} |\mathbb{D}v|^2 &= J|\mathbb{D}_{\mathcal{A}}v|^2 - (J-1)|\mathbb{D}v|^2 - J\left(|\mathbb{D}_{\mathcal{A}}v|^2 - |\mathbb{D}v|^2\right) \\ &= J|\mathbb{D}_{\mathcal{A}}v|^2 - (J-1)|\mathbb{D}v|^2 - J\left(\mathbb{D}_{\mathcal{A}}v + \mathbb{D}v\right) : \left(\mathbb{D}_{\mathcal{A}}v - \mathbb{D}v\right) \\ &=: I + II + III. \end{aligned} \tag{5.1.5}$$

Since the I and II terms are already in place, we just need to bound III. To do this, compute

$$\left(\mathbb{D}_{\mathcal{A}}v \pm \mathbb{D}v\right)_{ij} = \left(\mathcal{A} \pm I\right)_{ik} \partial_k v_j + \left(\mathcal{A} \pm I\right)_{jk} \partial_k v_i \tag{5.1.6}$$

and so

$$III = -J\left(\mathbb{D}_{\mathcal{A}}v + \mathbb{D}v\right) : \left(\mathbb{D}_{\mathcal{A}}v - \mathbb{D}v\right) \le \|J\|_{L^\infty}\|\mathcal{A}+I\|_{L^\infty}\|\mathcal{A}-I\|_{L^\infty}|\mathbb{D}v|^2. \tag{5.1.7}$$

The L^∞ norms can be bounded by universal constants by eq. (5.1.1) and eq. (5.1.2), so we conclude as desired. □

5.2 Estimates of the F forcing terms

We now present the estimates of the F forcing terms that appear in the geometric form of the equations eq. (4.1.1). Estimates of the same general form are now well-known in the literature: [GT13, TW14, JTW16, Tic18].

Theorem 5.2.1. *Let $F^{j,n}$ be defined by eq. (4.1.15), eq. (4.1.17), eq. (4.1.19), eq. (4.1.21). Assume that $\mathcal{E} \le \delta$ for the universal $\delta \in (0,1)$ given by proposition proposition 5.1.1. If $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$, then there exists a polynomial P with nonnegative universal coefficients such that*

$$\|F^{1,n}\|_0^2 + \|F^{2,n}\|_0^2 + \|\partial_t(JF^{2,n})\|_0^2 + \|F^{3,n}\|_0^2 + \|F^{4,n}\|_0^2 \lesssim P(\sigma)\mathcal{E}_n^0\mathcal{D}_n^\sigma \tag{5.2.1}$$

and

$$\|F^{2,n}\|_0^2 \lesssim \left(\mathcal{E}_n^0\right)^2. \tag{5.2.2}$$

Proof. The short version is, each of the nonlinearities is quadratic, i.e. is a product XY, where we can show that each is bounded by $\mathcal{E}$ and $\mathcal{D}$. We now record the details for each of the terms. First compute, using for instance [Les91],

$$\|\partial_t^n K\|_0^2 \lesssim \sum_{\ell=1}^{n} \left\|K^{\ell+1}\right\|_0^2 \left\|\partial_t^n\left(J^\ell\right)\right\|_0^2 \lesssim \mathcal{E}_n^0 \tag{5.2.3}$$

and

$$\|\partial_t^n \mathcal{A}_{jk}\|_0^2 \lesssim \|\partial_t^n(AK)\|_0^2 + \|\partial_t^n(BK)\|_0^2 + \|\partial_t^n K\|_0^2 \lesssim \sum_{0\le\ell\le n} \left\|\nabla\partial_t^\ell\hat{\eta}\right\|_0^2 \left\|\partial_t^{n-\ell}K\right\|_0^2 + \|\partial_t^n K\|_0^2 \lesssim \mathcal{E}_n^0. \tag{5.2.4}$$

Bounding $F^{1,n}$: Recall that

$$F_i^{1,n} = \sum_{0<\ell\leq n} C_{\ell n} \left[\partial_t^\ell \left(\partial_t \hat{\eta}\tilde{b}K\right) \partial_t^{n-\ell} \left(\partial_3 u_i\right) - \partial_t^\ell \left(u_j \mathcal{A}_{jk}\right) \partial_t^{n-\ell} \left(\partial_k u_i\right)\right] + \sum_{0<\ell\leq n} C_{\ell n} \left[-\partial_t^\ell \mathcal{A}_{jk} \partial_t^{n-\ell} \partial_k \left(S_{\mathcal{A}}(u,p)\right)_{ij} + \mathcal{A}_{jk}\mu \mathbb{D}_{\partial_t^\ell \mathcal{A}} \partial_t^{n-\ell} u\right] = 0. \tag{5.2.5}$$

We bound each of the four terms. The first term is bounded by

$$\left\|\partial_t^\ell \left(\partial_t \hat{\eta}\tilde{b}K\right) \partial_t^{n-\ell} \left(\partial_3 u_i\right)\right\|_0^2 \lesssim \left\|\partial_t^\ell \left(\partial_t \hat{\eta} K\right)\right\|_0^2 \mathcal{D}_n^\sigma \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma, \tag{5.2.6}$$

the second by

$$\left\|\partial_t^\ell \left(u_j \mathcal{A}_{jk}\right) \partial_t^{n-\ell} \left(\partial_k u_i\right)\right\|_0^2 \lesssim \left(\sum_{m=0}^{\ell} \left\|\partial_t^{\ell-m} u_j\right\|_0^2 \left\|\partial_t^m \mathcal{A}_{jk}\right\|_0^2\right) \left\|\partial_t^{n-\ell} u\right\|_1^2 \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma, \tag{5.2.7}$$

the third by

$$\left\|\partial_t^\ell \mathcal{A}_{jk} \partial_t^{n-\ell} \partial_k \left(S_{\mathcal{A}}(u,p)\right)_{ij}\right\|_0^2 \lesssim \left\|\partial_t^\ell \mathcal{A}_{jk}\right\|_0^2 \left\|\partial_t^{n-\ell} \partial_k p I - \partial_t^{n-\ell} \partial_k \left(\mathbb{D}_{\mathcal{A}} u\right)\right\|_0^2 \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma, \tag{5.2.8}$$

and the last by

$$\left\|\mathcal{A}_{jk}\mu \mathbb{D}_{\partial_t^\ell \mathcal{A}} \partial_t^{n-\ell} u\right\|_0^2 \lesssim \left\|\partial_t^\ell \mathcal{A}\right\|_0^2 \left\|\partial_t^{n-\ell} u\right\|_0^2 \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \tag{5.2.9}$$

Bounding $F^{2,n}$: We have that

$$\left\|F^{2,n}\right\|_0^2 = \left\|\sum_{0<\ell\leq n} C_{\ell n} \partial_t^\ell \mathcal{A}_{jk} \partial_t^{n-\ell} \partial_k u_j\right\|^2 \lesssim \sum_{0<\ell\leq n} \|\partial_t^\ell \mathcal{A}_{jk}\|_0^2 \|\partial_t^{n-\ell} \partial_k u_j\|_0^2 \lesssim \mathcal{E}_n^0 \min\{\mathcal{E}_n^0,\, \mathcal{D}_n^\sigma\}. \tag{5.2.10}$$

Bounding $\partial_t(JF^{2,n})$: We have that

$$\left\|\partial_t(JF^{2,n})\right\|_0^2 \lesssim \sum_{0<\ell\leq n} \left\|J\partial_t \left(\partial_t^\ell \mathcal{A}_{jk} \partial_t^{n-\ell} \partial_k u_j\right)\right\|_0^2 + \left\|\partial_t J \partial_t^\ell \mathcal{A}_{jk} \partial_t^{n-\ell} \partial_k u_j\right\|_0^2 \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \tag{5.2.11}$$

Bounding $F^{3,n}$: We have that

$$\|F^{3,n}\|_0^2 = \left\|\sum_{0<\ell\leq n} C_{\ell n} \partial_t^\ell u \cdot \partial_t^{n-\ell} \mathcal{N}\right\|_0^2 \lesssim \sum_{0<\ell\leq n} \|\partial_t^\ell u\|_0^2 \|\partial_t^{n-\ell} \nabla \eta\|_0^2 \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \tag{5.2.12}$$

Bounding $F^{4,n}$: Recall that

$$F_i^{4,n} = \sum_{0<\ell\leq n} C_{\ell n} \left[-\partial_t^{n-\ell} \left(S_{\mathcal{A}}(u,p)\right)_{ij} \partial_t^\ell \mathcal{N}_j + \left(\mu \mathbb{D}_{\partial_t^\ell \mathcal{A}} \partial_t^{n-\ell} u\right)_{ij} \mathcal{N}_j\right] + \sum_{0<\ell\leq n} C_{\ell n} \left(-\sigma \partial_t^{n-\ell} \mathfrak{H}(\eta) + g\partial_t^{n-\ell} \eta + \partial_t^{n-\ell} \left(A\omega^2 f''(\omega t)\eta\right)\right) \partial_t^{n-\ell} \mathcal{N}_i + \left(-\sigma \partial_t^n \left(\mathfrak{H}(\eta) - \Delta \eta\right)\right) \mathcal{N}_i. \tag{5.2.13}$$

The first term is bounded by

$$\left\|\partial_t^{n-\ell}\left(S_{\mathcal{A}}(u,p)\right)_{ij}\partial_t^\ell \mathcal{N}_j\right\|_0^2 \lesssim \left\|\partial_t^{n-\ell}pI-\partial_t^{n-\ell}\mathbb{D}_{\mathcal{A}}u\right\|_0^2\left\|\partial_t^\ell\nabla\eta\right\|_0^2 \lesssim \mathcal{E}_n^0\mathcal{D}_n^\sigma, \tag{5.2.14}$$

and the second term is bounded by

$$\left\|\left(\mu\mathbb{D}_{\partial_t^\ell\mathcal{A}}\partial_t^{n-\ell}u\right)_{ij}\mathcal{N}_j\right\|_0^2 \lesssim \|\partial_t^\ell\mathcal{A}\|_0^2\|\mathbb{D}\partial_t^{n-\ell}u\|_0^2 \lesssim \mathcal{E}_n^0\mathcal{D}_n^\sigma. \tag{5.2.15}$$

To bound the last two terms involving $\mathfrak{H}(\eta)$, we first compute

$$\begin{aligned}\mathfrak{H}(\eta) = \partial_i\left(\frac{\partial_i\eta}{\sqrt{1+|\nabla\eta|^2}}\right) &= \frac{\left(1+|\nabla\eta|^2\right)^{1/2}\partial_i^2\eta-\partial_i\eta\left(1+|\nabla\eta|^2\right)^{-1/2}\langle\nabla\eta,\partial_i\nabla\eta\rangle}{1+|\nabla\eta|^2}\\ &= \left(1+|\nabla\eta|^2\right)^{-1/2}\Delta\eta-\left(1+|\nabla\eta|^2\right)^{-3/2}\left(\partial_i\eta\langle\nabla\eta,\partial_i\nabla\eta\rangle\right).\end{aligned} \tag{5.2.16}$$

Then, the third term is bounded by

$$\begin{aligned}&\left\|\left(-\sigma\partial_t^{n-\ell}\mathfrak{H}(\eta)+g\partial_t^{n-\ell}\eta+\partial_t^{n-\ell}\left(A\omega^2f''(\omega t)\eta\right)\right)\partial_t^\ell\mathcal{N}_i\right\|_0^2\\ &\quad\lesssim P(\sigma)\left(\sum_{j=2}^{n+1}A\omega^j\right)^2\left(\sum_{m=0}^{n-\ell}\|\partial_t^m\partial_i\partial_j\eta\|_0^2+\|\partial_t^m\eta\|_0^2\right)\left\|\partial_t^\ell\nabla\eta\right\|_0^2 \lesssim P(\sigma)\mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.2.17}$$

For the fourth term, we first compute (using [Les91] and some computations in section appendix A.5.1)

$$\begin{aligned}\left\|\partial_t^n\left[\left(\frac{1}{\sqrt{1+|\nabla\eta|^2}}-1\right)\Delta\eta\right]\right\|_0^2 &\lesssim \sum_{\ell=1}^n\left\|\partial_t^\ell\frac{1}{\sqrt{1+|\nabla\eta|^2}}\right\|_0^2\left\|\Delta\partial_t^{n-\ell}\eta\right\|_0^2\\ &\qquad+\left\|\frac{\sqrt{1+|\nabla\eta|^2}-1}{\sqrt{1+|\nabla\eta|^2}}\right\|_0^2\|\Delta\partial_t^n\eta\|_0^2\\ &\lesssim \sum_{\ell=1}^n\sum_{m=1}^\ell\left\|\frac{1}{(1+|\nabla\eta|^2)^{(m+1)/2}}\right\|_0^2\left\|\partial_t^\ell\left(1+|\nabla\eta|^2\right)^{m/2}\right\|_0^2\left\|\partial_t^{n-\ell}\eta\right\|_2^2\\ &\qquad+\left\||\nabla\eta|^2\right\|_0^2\left\|\partial_t^n\eta\right\|_2^2\\ &\lesssim \sum_{\ell=0}^n\left\|\partial_t^\ell|\nabla\eta|^2\right\|_0^2\left\|\partial_t^{n-\ell}\eta\right\|_2^2 \lesssim \mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.2.18}$$

Then, using similar tricks as before,

$$\begin{aligned}\|-\sigma\partial_t^n\left(\mathfrak{H}(\eta)-\Delta\eta\right)\mathcal{N}_i\|_0^2 &\lesssim P(\sigma)\left\|\partial_t^n\left[\left(\frac{1}{\sqrt{1+|\nabla\eta|^2}}-1\right)\Delta\eta\right]\right\|_0^2+P(\sigma)\left\|\partial_t^n\frac{\partial_i\eta\langle\nabla\eta,\partial_i\nabla\eta\rangle}{(1+|\nabla\eta|^2)^{3/2}}\right\|_0^2\\ &\lesssim P(\sigma)\mathcal{E}_n^0\mathcal{D}_n^\sigma+P(\sigma)\sum_{\ell=0}^n\left\|\partial_t^\ell\left(\partial_i\eta\langle\nabla\eta,\partial_i\nabla\eta\rangle\right)\right\|_0^2\left\|\partial_t^{n-\ell}\frac{1}{(1+|\nabla\eta|^2)^{3/2}}\right\|_0^2\\ &\lesssim P(\sigma)\mathcal{E}_n^0\mathcal{D}_n^\sigma+P(\sigma)\mathcal{E}_n^0\sum_{\ell=0}^n\left\|\sum_{m=0}^{n-\ell}\left(\partial_t^m\frac{1}{\sqrt{1+|\nabla\eta|^2}}\right)\left(\partial_t^{n-\ell-m}\frac{1}{1+|\nabla\eta|^2}\right)\right\|_0^2\\ &\lesssim P(\sigma)\mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.2.19}$$

$\square$

5.3 Estimates of the G forcing terms

We now present the estimates for the G^i nonlinearities. Define

$$\mathcal{Y}_n := \sum_{j=0}^{n-1} \|\partial_t^j G^1\|_{2n-2j-1}^2 + \|\partial_t^j G^2\|_{2n-2j}^2 + \left\|\partial_t^j G^4\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 + \sum_{j=2}^{n} \left\|\partial_t^j G^3\right\|_{H^{2n-2j+1/2}(\Sigma)}^2$$
$$+ \left\|G^3\right\|_{H^{2n-1}(\Sigma)}^2 + \left\|\partial_t G^3\right\|_{H^{2n-2}(\Sigma)}^2 + \sigma^2 \left(\left\|G^3\right\|_{H^{2n+1/2}(\Sigma)}^2 + \left\|\partial_t G^3\right\|_{H^{2n-3/2}(\Sigma)}^2 \right). \tag{5.3.1}$$

and

$$\mathcal{W}_n := \sum_{j=0}^{n-1} \left\|\partial_t^j G^1\right\|_{2n-2j-2}^2 + \left\|\partial_t^j G^2\right\|_{2n-2j-1}^2 + \left\|\partial_t^j G^3\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 + \left\|\partial_t^j G^4\right\|_{H^{2n-2j-3/2}(\Sigma)}^2. \tag{5.3.2}$$

These nonlinearities are the ones generated by elliptic regularity estimates.

Theorem 5.3.1. *Let G^i be defined by eq. (4.2.12), eq. (4.2.13), eq. (4.2.15), and eq. (4.2.16). Assume that $\mathcal{E} \le \delta$ for the universal $\delta \in (0,1)$ given by proposition proposition 5.1.1. If $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$, then there exists a polynomial P with nonnegative universal coefficients such that*

$$\mathcal{Y}_n \lesssim P(\sigma) \left(\mathcal{E}_n^0 \mathcal{D}_n^\sigma + \mathcal{K}\mathcal{F}_n \right) \tag{5.3.3}$$

and

$$\mathcal{W}_n \lesssim P(\sigma) \left(\mathcal{E}_n^0 \mathcal{E}_n^\sigma + \mathcal{K}\mathcal{F}_n \right). \tag{5.3.4}$$

Furthermore, in the case that $n = 1$ and fixed $\sigma > 0$, we have

$$\mathcal{W}_1 \lesssim P(\sigma)(\mathcal{E}_1^\sigma)^2. \tag{5.3.5}$$

and

$$\|G^1\|_1^2 + \|G^2\|_2^2 + \|G^3\|_{5/2}^2 + \|\partial_t G^3\|_{1/2}^2 + \|G^4\|_1^2 \lesssim P(\sigma)\mathcal{E}_1^\sigma \mathcal{D}_1^\sigma. \tag{5.3.6}$$

Note the above is just $\mathcal{Y}_1$ after considering σ as a fixed constant, with $\|G^4\|_1$ instead of $\|G^4\|_{3/2}$.

Proof. We proceed to tackle each of these estimates.

Bounding G^1: We first remark that throughout this proof we will be bounding $\tilde{b}$, K, A, and B above by constants via proposition 5.1.1.

Recall that

$$G^1 = \partial_t \hat{\eta} \tilde{b} K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u - \operatorname{div} \mathbb{D}_{I-\mathcal{A}} u - \operatorname{div}_{\mathcal{A}-I} (pI - \mathbb{D}_{\mathcal{A}} u). \tag{5.3.7}$$

We bound each of the terms above. Let $j \in \{0, 1, \ldots, n-1\}$.

- **First term:** We first work on bounding j temporal derivatives of $\partial_t \hat{\eta} \tilde{b} K \partial_3 u$ in $H^{2n-2j-1}(\Omega)$ by $\mathcal{E}_n^0 \mathcal{D}_n^\sigma$. When $j \le n-2$, we have that $2n - 2j - 1 \ge 3 > 3/2$ so we may use Sobolev product estimates to bound

$$\begin{aligned}\left\|\partial_t^j\left(\partial_t\hat{\eta}\tilde{b}K\partial_3 u\right)\right\|_{2n-2j-1}^2 &\lesssim \left(\sum_{m=0}^{j}\left\|\partial_t^{m+1}\hat{\eta}\right\|_{2n-2j-1}^2\right)\left(\sum_{m=0}^{j}\|\partial_t^m K\|_{2n-2j-1}^2\right)\left(\sum_{m=0}^{j}\|\partial_t^m u\|_{2n-2j}^2\right)\\ &\lesssim \left(\sum_{m=1}^{j+1}\|\partial_t^m\eta\|_{2n-2j-3/2}^2\right)\left(\sum_{m=0}^{j}\sum_{\ell=0}^{m}\left\|\partial_t^m\hat{\eta}^\ell\right\|_{2n-2j}^2\right)\left(\sum_{m=0}^{j}\|\partial_t^m u\|_{2n-2j}^2\right)\\ &\lesssim \left(\sum_{m=0}^{j}\sum_{\ell=1}^{m}\left\|\partial_t^m\eta^\ell\right\|_{2n-2j-1/2}^2\right)\mathcal{E}_n^0\min\left\{\mathcal{E}_n^0,\mathcal{D}_n^0\right\}\\ &\lesssim \mathcal{E}_n^0\mathcal{E}_n^0\min\left\{\mathcal{E}_n^0,\mathcal{D}_n^0\right\}\lesssim\mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.3.8}$$

When $j=n-1$, we have $2n-2j-1=1$ so

$$\begin{aligned}\left\|\partial_t^{n-1}\left(\partial_t\hat{\eta}\tilde{b}K\partial_3 u\right)\right\|_1^2 &\lesssim \left\|\partial_t^{n-1}\left(\partial_t\hat{\eta}\tilde{b}K\partial_3 u\right)\right\|_0^2+\sum_{i=1}^{3}\left\|\partial_i\partial_t^{n-1}\left(\partial_t\hat{\eta}\tilde{b}K\partial_3 u\right)\right\|_0^2\\ &\lesssim \left(\sum_{m=0}^{n-1}\left\|\partial_t^{m+1}\hat{\eta}\right\|_1^2\right)\left(\sum_{m=0}^{n-1}\|\partial_t^m K\|_1^2\right)\left(\sum_{m=0}^{n-1}\|\partial_t^m u\|_2^2\right)\\ &\lesssim \mathcal{E}_n^0\mathcal{E}_n^0\min\left\{\mathcal{E}_n^0,\mathcal{D}_n^0\right\}\lesssim\mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.3.9}$$

We easily see that the same computations apply for the $\mathcal{W}_n$ bound.

- **Second term:** Recall that $u\cdot\nabla_{\mathcal{A}}u=u_i\mathcal{A}_{ij}\partial_j u$. We first bound j temporal derivatives of $u\cdot\nabla_{\mathcal{A}}u$ in $H^{2n-2j-1}(\Omega)$ by $\mathcal{E}_n^0\mathcal{D}_n^\sigma$. When $j\le n-2$, we have that $2n-2j-1\ge 3>3/2$ so we use Sobolev product estimates

$$\begin{aligned}\left\|\partial_t^j\left(u\cdot\nabla_{\mathcal{A}}u\right)\right\|_{2n-2j-1}^2 &\lesssim \left(\sum_{m=0}^{j}\|\partial_t^m u\|_{2n-2j-1}^2\right)\left(\sum_{m=0}^{j}\|\partial_t^m\mathcal{A}\|_{2n-2j-1}^2\right)\left(\sum_{m=0}^{j}\|\partial_t^m u\|_{2n-2j}^2\right)\\ &\lesssim \mathcal{E}_n^0\mathcal{E}_n^0\left(\sum_{m=0}^{j}\sum_{\ell=1}^{m}\left\|\partial_t^m\hat{\eta}^\ell\right\|_{2n-2j}^2\right)\lesssim\mathcal{E}_n^0\mathcal{E}_n^0\left(\sum_{m=0}^{j}\sum_{\ell=1}^{m}\left\|\partial_t^m\eta^\ell\right\|_{2n-2j-1/2}^2\right)\\ &\lesssim \mathcal{E}_n^0\mathcal{E}_n^0\min\left\{\mathcal{E}_n^0,\mathcal{D}_n^0\right\}\lesssim\mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.3.10}$$

When $j=n-1$, we have $2n-2j-1=1$ so

$$\begin{aligned}\left\|\partial_t^{n-1}\left(u\cdot\nabla_{\mathcal{A}}u\right)\right\|_1^2 &\lesssim \left\|\partial_t^{n-1}\left(u\cdot\nabla_{\mathcal{A}}u\right)\right\|_0^2+\sum_{i=1}^{3}\left\|\partial_i\partial_t^{n-1}\left(u\cdot\nabla_{\mathcal{A}}u\right)\right\|_0^2\\ &\lesssim \left(\sum_{m=0}^{n-1}\|\partial_t^m u\|_2^2\right)\left(\sum_{m=0}^{n-1}\|\partial_t^m\mathcal{A}\|_1^2\right)\lesssim\mathcal{E}_n^0\mathcal{E}_n^0\min\left\{\mathcal{E}_n^0,\mathcal{D}_n^0\right\}\lesssim\mathcal{E}_n^0\mathcal{D}_n^\sigma.\end{aligned} \tag{5.3.11}$$

Again, the above computations also apply for the $\mathcal{W}_n$ bound.

- **Third term:** We first work on bounding j temporal derivatives of $\operatorname{div}_{I-\mathcal{A}}u$ in $H^{2n-2j-1}(\Omega)$ by $\mathcal{E}_n^0\mathcal{D}_n^\sigma$. Again, we split into $j\le n-2$, for which Sobolev product estimates apply, and

$j = n-1$. We have

$$
\begin{aligned}
\left\|\partial_t^j \left(\operatorname{div} \mathbb{D}_{I-\mathcal{A}} u\right)_i\right\|_{2n-2j-1}^2 &\lesssim \left\|\partial_t^j \left[\partial_\ell \left((I-\mathcal{A})_{im} \partial_m u_\ell + (I-\mathcal{A})_{\ell m} \partial_m u_i\right)\right]\right\|_{2n-2j-1}^2 \\
&= \left\|\partial_t^j \left[\partial_\ell \left((I-\mathcal{A})_{i3} \partial_3 u_\ell + (I-\mathcal{A})_{\ell 3} \partial_3 u_i\right)\right]\right\|_{2n-2j-1}^2 \\
&\lesssim \left\|\partial_t^j \left[\nabla (I-\mathcal{A})_{\ell 3} \partial_3 u_m\right]\right\|_{2n-2j-1}^2 + \left\|\partial_t^j \left[(I-\mathcal{A})_{\ell 3} \nabla \partial_3 u_m\right]\right\|_{2n-2j-1}^2 \\
&\lesssim \left(\sum_{m=0}^{j} \|\partial_t^m (I-\mathcal{A})_{\ell 3}\|_{2n-2j}^2\right) \left(\sum_{m=0}^{j} \|\partial_t^m u\|_{2n-2j+1}^2\right) \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma
\end{aligned} \tag{5.3.12}
$$

since the third column of $I-\mathcal{A}$ is either just linear or quadratic in derivatives of $\hat{\eta}$ that we control in $\mathcal{E}_n^0$. We can't simultaneously use $\mathcal{E}_n^0$ for the u term this time since we only control $\partial_t^j u$ in $H^{2n-2j}(\Omega)$ in $\mathcal{E}_n^0$. For $j = n-1$, similarly to the above, we have

$$
\begin{aligned}
\left\|\partial_t^{n-1} \left(\operatorname{div} \mathbb{D}_{I-\mathcal{A}} u\right)_i\right\|_1^2 &\lesssim \left\|\partial_t^{n-1} \left(\operatorname{div} \mathbb{D}_{I-\mathcal{A}} u\right)_i\right\|_0^2 + \left\|\partial_t^{n-1} \nabla \left(\operatorname{div} \mathbb{D}_{I-\mathcal{A}} u\right)_i\right\|_0^2 \\
&\lesssim \left(\sum_{m=0}^{n-1} \|\partial_t^m (I-\mathcal{A})_{\ell 3}\|_2^2\right) \left(\sum_{m=0}^{n-1} \|\partial_t^m u\|_3^2\right) \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma.
\end{aligned} \tag{5.3.13}
$$

Next, we bound j temporal derivatives of $\operatorname{div}_{I-\mathcal{A}} u$ in $H^{2n-2j-2}(\Omega)$ by $\mathcal{E}_n^0 \mathcal{E}_n^\sigma$. The computation for $j \le n-3$ is almost exactly the same as the above, except we need to control one less derivative on u and thus we get to control the u term by $\mathcal{E}_n^\sigma$ rather than $\mathcal{D}_n^\sigma$. The same is true for $j = n-2$ and $j = n-1$, so we omit in the interest of brevity.

- **Fourth term:** The term that we need to bound is

$$
\operatorname{div}_{\mathcal{A}-I}(pI - \mathbb{D}_{\mathcal{A}} u) = \operatorname{div}_{\mathcal{A}-I}(pI) - \operatorname{div}_{\mathcal{A}-I}(\mathbb{D}_{\mathcal{A}} u). \tag{5.3.14}
$$

We bound j temporal derivatives of the above in $H^{2n-2j-1}(\Omega)$. We have

$$
\begin{aligned}
&\left\|\partial_t^j \operatorname{div}_{\mathcal{A}-I}(pI)\right\|_{2n-2j-1}^2 = \left\|\partial_t^j \left[(\mathcal{A}-I)_{\ell m} \partial_m (pI)_{i\ell} e_i\right]\right\|_{2n-2j-1}^2 = \left\|\partial_t^j \left[(\mathcal{A}-I)_{i3} \partial_3 p e_i\right]\right\|_{2n-2j-1}^2 \\
&\quad\lesssim \sum_{\substack{\alpha \in \mathbb{N}^2 \\ |\alpha| \le 2n-2j-1}} \left\|\partial_t^j \partial^\alpha \left[(\mathcal{A}-I)_{i3} \partial_3 p e_i\right]\right\|_0^2 \\
&\quad\lesssim \sum_{\substack{\alpha \in \mathbb{N}^2 \\ |\alpha| \le 2n-2j-1}} \sum_{m=0}^{j} \sum_{\beta+\gamma=\alpha} \left\|\partial^m \partial^\beta (\mathcal{A}-I)_{i3} \partial^{j-m} \partial^\gamma \partial_3 p e_i\right\|_0^2 \\
&\quad\lesssim \sum_{\substack{\alpha \in \mathbb{N}^2 \\ |\alpha| \le 2n-2j-1}} \sum_{m=0}^{j} \sum_{\beta+\gamma=\alpha} \|\partial^m (\mathcal{A}-I)_{i3}\|_{|\beta|}^2 \left\|\partial^{j-m} \partial_3 p e_i\right\|_{|\gamma|}^2 \\
&\quad\lesssim \sum_{m=0}^{j} \|\partial^m (\mathcal{A}-I)_{i3}\|_{2n-2j-1}^2 \left\|\partial^{j-m} p\right\|_{2n-2j}^2 \lesssim \sum_{m=0}^{j} \sum_{\ell=1}^{m} \left\|\partial^m \hat{\eta}^\ell\right\|_{2n-2j}^2 \left\|\partial^{j-m} p\right\|_{2n-2(j-m)}^2 \\
&\quad\lesssim \sum_{m=0}^{j} \sum_{\ell=1}^{m} \left\|\partial^m \eta^\ell\right\|_{2n-2j-1/2}^2 \left\|\partial^{j-m} p\right\|_{2n-2(j-m)}^2 \lesssim \mathcal{E}_n^0 \min\left\{\mathcal{E}_n^0, \mathcal{D}_n^0\right\} \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma.
\end{aligned} \tag{5.3.15}
$$

The computations for this term apply for bounding the above in $H^{2n-2j-2}(\Omega)$ by $\mathcal{E}_n^0\mathcal{E}_n^\sigma$. Next, we work on $\operatorname{div}_{\mathcal{A}-I}(\mathbb{D}_{\mathcal{A}}u)$. This term needs $(2n-2j-1)+2 = 2n-2j+1$ derivatives for u (one extra for divergence, one extra for symmetric gradient) so we need to handle these separately for $\mathcal{D}_n^\sigma$ and $\mathcal{E}_n^\sigma$, but the computations are similar so we only do one of these. We have

$$
\begin{aligned}
\left\|\partial_t^j \operatorname{div}_{\mathcal{A}-I}(\mathbb{D}_{\mathcal{A}}u)\right\|_{2n-2j-1}^2 &= \left\|\partial_t^j\left[(\mathcal{A}-I)_{\ell m}\partial_m(\mathbb{D}_{\mathcal{A}}u)_{i\ell}e_i\right]\right\|_{2n-2j-1}^2 \\
&= \left\|\partial_t^j\left[(\mathcal{A}-I)_{\ell m}\partial_m(\mathcal{A}_{ik}\partial_k u_\ell + \mathcal{A}_{\ell k}\partial_k u_i)\right]\right\|_{2n-2j-1}^2 \\
&\lesssim \sum_{p+q+r=j}\ \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}^2\\ |\alpha|+|\beta|+|\gamma|\le 2n-2j-1}} \left\|\partial_t^p\partial^\alpha(\mathcal{A}-I)_{\ell m}\partial_m(\partial_t^q\partial^\beta\mathcal{A}_{ik}\partial_k\partial_t^r\partial^\gamma u_\ell)\right\|_0^2 \\
&\qquad + \left\|\partial_t^p\partial^\alpha(\mathcal{A}-I)_{\ell m}\partial_m(\partial_t^q\partial^\beta\mathcal{A}_{\ell k}\partial_k\partial_t^r\partial^\gamma u_i)\right\|_0^2 \\
&\lesssim \sum_{p+q+r=j}\ \sum_{\substack{\alpha,\beta,\gamma\in\mathbb{N}^2\\ |\alpha|+|\beta|+|\gamma|\le 2n-2j-1}} \left\|\partial_t^p\partial^\alpha(\mathcal{A}-I)_{\ell m}\right\|_0^2 \\
&\qquad \left(\left\|\partial_m(\partial_t^q\partial^\beta\mathcal{A}_{ik}\partial_k\partial_t^r\partial^\gamma u_\ell)\right\|_0^2 + \left\|\partial_m(\partial_t^q\partial^\beta\mathcal{A}_{\ell k}\partial_k\partial_t^r\partial^\gamma u_i)\right\|_0^2\right) \\
&\lesssim \sum_{p+q+r=j}\sum_{m=0}^{p}\left\|\partial_t^p\eta^m\right\|_{2n-2j-1/2}^2\left\|\partial_t^q\mathcal{A}_{ik}\right\|_{2n-2j}^2\left\|\partial_t^r u\right\|_{2n-2j+1}^2 \\
&\lesssim \sum_{p+q+r=j}\left(\sum_{m=1}^{p}\left\|\partial_t^p\eta^m\right\|_{2n-2j-1/2}^2\right)\left(\sum_{\ell=1}^{q}\left\|\partial_t^q\eta^\ell\right\|_{2n-2j+1/2}^2\right)\mathcal{D}_n^\sigma \lesssim \mathcal{E}_n^0\mathcal{E}_n^0\mathcal{D}_n^\sigma \lesssim \mathcal{E}_n^0\mathcal{D}_n^\sigma.
\end{aligned}
\tag{5.3.16}
$$

Note that the second term above is fine since we take at least one time derivative and hence we can control it with $\mathcal{E}_n^0$ – the only time we worry is if we take no time derivatives of η.

Bounding G^2: We first show for $\mathcal{W}_n$, since Recall that

$$
G^2 = \operatorname{div}_{I-\mathcal{A}} u. \tag{5.3.17}
$$

Now write

$$
\operatorname{div}_{I-\mathcal{A}} u = (I-\mathcal{A})_{ik}\partial_k u_i = (AK)\partial_3 u_1 + (BK)\partial_3 u_2 + (1-K)\partial_3 u_3. \tag{5.3.18}
$$

We handle just the A term since the proof for the other two terms is similar. Write using the Leibniz rule that

$$
\|\partial_t^j(AK\partial_3 u_1)\|_{2n-2j-1} \lesssim \sum_{\ell=0}^{j}\|\partial_t^\ell(AK)\partial_t^{j-\ell}(\partial_3 u_1)\|_{2n-2j-1}. \tag{5.3.19}
$$

If $j \le n-2$, we may use similar tactics as in eq. (5.2.4) using Sobolev product estimates to bound the AK terms by $\mathcal{E}_n^0$; this leaves the u terms to be bounded by $\mathcal{E}_n^\sigma$. For the particulars, recall that

$$
\left\|\partial_t^k K\right\|_{2n-2j-1}^2 \lesssim \sum_{\ell=0}^{k}\left\|K^{\ell+1}\partial_t^k(J^\ell)\right\|_{2n-2j-1}^2 \lesssim \sum_{\ell=0}^{k}\left\|K\right\|_{2n-2j-1}^{2(\ell+1)}\left\|\partial_t^k(J^\ell)\right\|_{2n-2j-1}^2. \tag{5.3.20}
$$

The tricky part lies in bounding $\partial_t^k(J^\ell)$ in $2n-2j-1$ derivatives. To do this, write

$$\begin{aligned}\partial_t^k(J^\ell) &= \partial_t^k\left[\left(1+\frac{\hat\eta}{b}+\partial_3\hat\eta\tilde b\right)\cdots\left(1+\frac{\hat\eta}{b}+\partial_3\hat\eta\tilde b\right)\right]\\ &= \sum_{k_1+\cdots+k_\ell=k}\partial_t^{k_1}\left(1+\frac{\hat\eta}{b}+\partial_3\hat\eta\tilde b\right)\cdots\partial_t^{k_\ell}\left(1+\frac{\hat\eta}{b}+\partial_3\hat\eta\tilde b\right).\end{aligned} \tag{5.3.21}$$

Now when we take the $H^{2n-2j-1}$ norm of both sides, the highest regularity term we need to control is $\|\partial_t^k\hat\eta\|_{2n-2j}\lesssim\mathcal{E}_n^0$. Combining this with $\|K\|_{2n-2j-1}\lesssim\mathcal{E}_n^\sigma$ gives the desired bound.

If $j=n-1$, then we instead write

$$\sum_{\ell=0}^{n-1}\|\partial_t^\ell(AK)\partial_t^{n-1-\ell}(\partial_3 u_1)\|_1\lesssim\sum_{\ell=0}^{n-1}\|\partial_t^\ell(AK)\partial_t^{n-1-\ell}u\|_0+\|\nabla[\partial_t^\ell(AK)\partial_t^{n-1-\ell}u]\|_0\lesssim\mathcal{E}_n^0\mathcal{E}_n^\sigma. \tag{5.3.22}$$

A similar proof works for bounding by $\mathcal{D}_n^\sigma$, since the highest order derivative of $\partial_t^{n-1}\eta$ that we require is 1/2.

Bounding G^3: Write

$$G^3=u\cdot(\mathcal{N}-e_3)=u\cdot(-\partial_1\eta,-\partial_2\eta,0)=-\partial_1\eta u_1-\partial_2\eta u_2. \tag{5.3.23}$$

We'll handle the first-coordinate terms here; the second-coordinate terms are identical.

- $\mathcal{W}_n$ **bound:**

  We first deal with the $\mathcal{E}_n^0\mathcal{E}_n^\sigma$ bound, for which $0\le j\le n-1$. Consider the case of $j=0$. If $n\ge 2$, then

$$\begin{aligned}\|\partial_1\eta u_1\|^2_{H^{2n-1/2}(\Sigma)} &\lesssim \|\partial_1\eta\|^2_{H^{2n-1/2}(\Sigma)}\|u_1\|^2_{L^\infty(\Sigma)}+\|\partial_1\eta\|^2_{L^\infty(\Sigma)}\|u_1\|^2_{H^{2n-1/2}(\Sigma)}\\ &\lesssim \|\eta\|^2_{2n+1/2}\|u_1\|^2_{C^1(\Sigma)}+\|\eta\|^2_{C^1(\Sigma)}\|u\|^2_{2n}\lesssim\|\eta\|^2_{2n+1/2}\|u\|^2_{H^3(\Sigma)}+\|\eta\|_3^2\|u\|^2_{2n}\\ &\lesssim \|\eta\|^2_{2n+1/2}\|u\|^2_{H^3(\Sigma)}+\|\eta\|^2_{2n}\|u\|^2_{2n}\lesssim\mathcal{K}\mathcal{F}_n+\left(\mathcal{E}_n^0\right)^2\end{aligned} \tag{5.3.24}$$

  and if $n=1$, then

$$\|\partial_1\eta u_1\|^2_{H^{3/2}(\Sigma)}\lesssim\|\partial_1\eta\|^2_{H^{3/2}(\Sigma)}\|u_1\|^2_{H^{3/2}(\Sigma)}\lesssim\|\eta\|^2_{5/2}\|u_1\|_2^2\lesssim\left(\mathcal{E}_n^0\right)^2. \tag{5.3.25}$$

  We now consider $1\le j\le n-1$. Using the Leibniz rule and Sobolev product estimates of theorem C.1.1 since $2n-2\ell-\frac12\ge 2n-2(n-1)-\frac12=\frac32>1$, we have

$$\begin{aligned}\left\|\partial_t^j(-\partial_1\eta u_1)\right\|^2_{H^{2n-2j-1/2}(\Sigma)} &\le\sum_{0\le\ell\le j}\left\|\partial_1(\partial_t^\ell\eta)\partial_t^{j-\ell}u_1\right\|^2_{H^{2n-2j-1/2}(\Sigma)}\\ &\lesssim\sum_{0\le\ell\le j}\|\partial_t^\ell\eta\|^2_{H^{2n-2j+1/2}(\Sigma)}\|\partial_t^{j-\ell}u_1\|^2_{H^{2n-2j-1/2}(\Sigma)}\\ &\lesssim\|\eta\|^2_{2n-3/2}\|\partial_t^j u_1\|^2_{2n-2j}+\sum_{1\le\ell\le j}\|\partial_t^\ell\eta\|^2_{2n-2\ell+1/2}\|\partial_t^{j-\ell}u\|^2_{2n-2j}\lesssim\left(\mathcal{E}_1^0\right)^2.\end{aligned} \tag{5.3.26}$$

- $\mathcal{Y}_n$ **bound:**

– $\sum_{j=2}^{n} \|\partial_t^j G^3\|^2_{H^{2n-2j+1/2}(\Sigma)}$ **bound:**

We now deal with the $\mathcal{E}_n^0 \mathcal{D}_n^\sigma$ term which is not bound to σ, for $2 \le j \le n$ now. First assume $j \le n-1$. Because $j \le n-1$, $2n - 2j + \frac{1}{2} > 1$, so we may write

$$
\begin{aligned}
\left\|\partial_t^j(-\partial_1 \eta u_1)\right\|^2_{H^{2n-2j+1/2}(\Sigma)} &\le \sum_{0\le\ell\le j} \left\|\partial_1(\partial_t^\ell \eta)\partial_t^{j-\ell} u_1\right\|^2_{H^{2n-2j+1/2}(\Sigma)} \\
&\lesssim \|\partial_1 \eta \partial_t^j u_1\|^2_{H^{2n-2j+1/2}(\Sigma)} + \sum_{1\le\ell\le j} \|\partial_1(\partial_t^\ell \eta)\|^2_{H^{2n-2j+1/2}(\Sigma)} \|\partial_t^{j-\ell} u_1\|^2_{H^{2n-2(j-\ell)+1/2}(\Sigma)} \\
&\lesssim \|\eta\|^2_{H^{2n-5/2}(\Sigma)} \|\partial_t^j u_1\|^2_{H^{2n-2j+1/2}(\Sigma)} + \sum_{1\le\ell\le j} \|\partial_t^\ell \eta\|^2_{H^{2n-2\ell+3/2}(\Sigma)} \|\partial_t^{j-\ell} u_1\|^2_{H^{2n-2(j-\ell)+1/2}(\Sigma)} \\
&\lesssim \|\eta\|^2_{2n-5/2} \|\partial_t^j u\|^2_{2n-2j+1} + \sum_{1\le\ell\le j} \|\partial_t^\ell \eta\|^2_{2n-2\ell+3/2} \|\partial_t^{j-\ell} u\|^2_{2n-2(j-\ell)+1} \lesssim \mathcal{E}_n^0 \mathcal{D}_n^0.
\end{aligned}
\tag{5.3.27}
$$

Now consider the $j = n$ case. We apply the Leibniz rule as before and estimate the terms with $\ell \ge 2$ by using lemma A.2 of [GT13]:

$$
\begin{aligned}
\sum_{\ell=2}^{n} \left\|(\partial_1 \partial_t^\ell \eta)(\partial_t^{n-\ell} u_1)\right\|^2_{H^{1/2}(\Sigma)} &\lesssim \sum_{\ell=2}^{n} \|\partial_1 \partial_t^\ell \eta\|^2_{1/2} \|\partial_t^{n-\ell} u_1\|^2_{C^1(\Sigma)} \lesssim \sum_{\ell=2}^{n} \|\partial_t^\ell \eta\|_{3/2} \|\partial_t^{n-\ell} u\|^2_{H^3(\Sigma)} \\
&\lesssim \sum_{\ell=2}^{n} \|\partial_t^\ell \eta\|_{3/2} \|\partial_t^{n-\ell} u\|^2_4 \\
&\lesssim \sum_{\ell=2}^{n} \|\partial_t^\ell \eta\|_{2n-2\ell+3/2} \|\partial_t^{n-\ell} u\|^2_{2n-2(n-\ell)+1} \lesssim \mathcal{E}_n^0 \mathcal{D}_n^0.
\end{aligned}
\tag{5.3.28}
$$

For the $\ell \in \{0, 1\}$ terms, we use the Poisson extension to compute

$$
\begin{aligned}
\sum_{\ell=0}^{1} \left\|(\partial_1 \partial_t^\ell \eta)(\partial_t^{n-\ell} u_1)\right\|^2_{H^{1/2}(\Sigma)} &\lesssim \sum_{\ell=0}^{1} \left\|(\partial_1 \partial_t^\ell \hat\eta)(\partial_t^{n-\ell} u_1)\right\|^2_1 \\
&\lesssim \sum_{\ell=0}^{1} \left\|(\partial_1 \partial_t^\ell \hat\eta)(\partial_t^{n-\ell} u_1)\right\|^2_0 + \left\|\nabla(\partial_1 \partial_t^\ell \hat\eta)(\partial_t^{n-\ell} u_1)\right\|^2_0 + \left\|(\partial_1 \partial_t^\ell \hat\eta)\nabla(\partial_t^{n-\ell} u_1)\right\|^2_0 \\
&\lesssim \sum_{\ell=0}^{1} \|\partial_t^\ell \hat\eta\|^2_2 \|\partial_t^{n-\ell} u\|^2_0 + \|\partial_t^\ell \hat\eta\|^2_1 \|\partial_t^{n-\ell} u\|^2_1 \\
&\lesssim \sum_{\ell=0}^{1} \|\partial_t^\ell \eta\|^2_{3/2} \|\partial_t^{n-\ell} u\|^2_0 + \|\partial_t^\ell \eta\|^2_{1/2} \|\partial_t^{n-\ell} u\|^2_1 \\
&\lesssim \|\eta\|^2_{3/2} \|\partial_t^n u\|^2_1 + \|\partial_t \eta\|^2_{3/2} \|\partial_t^{n-1} u\|^2_1 \lesssim \mathcal{E}_n^0 \mathcal{D}_n^0.
\end{aligned}
\tag{5.3.29}
$$

– $\|G^3\|_{H^{2n-1}(\Sigma)}$ **bound:**

Now we deal with $\|G^3\|_{H^{2n-1}(\Sigma)}$. Fortunately, this one is easy: assuming $n \ge 2$, write

$$
\|\partial_1 \eta u_1\|^2_{H^{2n-1}(\Sigma)} \lesssim \|\partial_1 \eta\|^2_{H^{2n-1}(\Sigma)} \|u_1\|^2_{H^{2n-1}(\Sigma)} \lesssim \|\eta\|^2_{2n} \|u\|^2_{2n-1/2} \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \tag{5.3.30}
$$

If $n = 1$, we instead write

$$\begin{aligned} \|G^3\|_{H^1(\Sigma)} &\lesssim \|\partial_1 \eta u_1\|^2_{H^0(\Sigma)} + \|\nabla \partial_1 \eta u_1\|^2_{H^0(\Sigma)} + \|\partial_1 \eta \nabla u_1\|^2_{H^0(\Sigma)} \\ &\lesssim \|\eta\|_2^2 \|u\|^2_{H^0(\Sigma)} + \|\eta\|_1^2 \|u\|^2_{H^1(\Sigma)} \lesssim \|\eta\|_2^2 \|u\|_2^2 \lesssim \mathcal{E}_1^0 \mathcal{D}_1^0. \end{aligned} \tag{5.3.31}$$

– $\|\partial_t G^3\|_{H^{2n-2}(\Sigma)}$ **bound:**
Next, the $\|\partial_t G^3\|_{H^{2n-2}(\Sigma)}$ term. If $n \geq 2$ then Sobolev product estimates apply and if $n = 1$ then we simply have Cauchy-Schwarz, so

$$\begin{aligned} \|\partial_t(\partial_1 \eta u_1)\|^2_{H^{2n-2}(\Sigma)} &\lesssim \|\partial_1(\partial_t \eta) u_1\|^2_{H^{2n-2}(\Sigma)} + \|\partial_1 \eta \partial_t u_1\|^2_{H^{2n-2}(\Sigma)} \\ &\lesssim \|\partial_1 \partial_t \eta\|^2_{H^{2n-2}(\Sigma)} \|u_1\|^2_{H^{2n-2}(\Sigma)} + \|\partial_1 \eta\|^2_{H^{2n-2}(\Sigma)} \|\partial_t u_1\|^2_{H^{2n-2}(\Sigma)} \\ &\lesssim \|\partial_t \eta\|^2_{H^{2n-1}(\Sigma)} \|u\|^2_{2n-3/2} + \|\eta\|^2_{H^{2n-1}(\Sigma)} \|\partial_t u\|^2_{2n-3/2} \\ &\lesssim \mathcal{D}_n^\sigma \mathcal{E}_n^0 + \mathcal{E}_n^0 \mathcal{D}_n^\sigma \lesssim \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \end{aligned} \tag{5.3.32}$$

– $\sigma^2 \|G^3\|_{H^{2n+1/2}(\Sigma)}$ **bound:**
Next, the $\sigma^2 \|G^3\|_{H^{2n+1/2}(\Sigma)}$ term. We'll actually control this one a little differently, because the previous techniques do not actually work. The key point is the estimate

$$\|G^3\|^2_{H^{2n+1/2}(\Sigma)} \lesssim \|\nabla^{2n} G^3\|^2_{H^{1/2}(\Sigma)} + \|G^3\|^2_{H^{1/2}(\Sigma)}. \tag{5.3.33}$$

The $H^{1/2}$ term can be bounded in the same way that we've done before, so it remains to bound the gradient term. To do this, write

$$\begin{aligned} \sigma^2 \|\nabla^{2n} G^3\|^2_{H^{1/2}(\Sigma)} &\lesssim \sum_{j=0}^{2n} \sigma^2 \|\nabla^{j+1} \hat{\eta} \nabla^{2n-j} u\|_1^2 \lesssim \sigma^2 \|\hat{\eta}\|_2^2 \|u\|^2_{2n+1} + \sum_{j=1}^{2n} \sigma^2 \|\hat{\eta}\|^2_{j+2} \|u\|^2_{2n-j+1} \\ &\lesssim \sigma^2 \|\eta\|^2_{3/2} \|u\|^2_{2n+1} + \sigma^2 \|\eta\|^2_{2n+3/2} \|u\|^2_{2n} \lesssim \sigma^2 \mathcal{E}_n^0 \mathcal{D}_n^0 + \mathcal{D}_n^\sigma \mathcal{E}_n^0 \lesssim P(\sigma) \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \end{aligned} \tag{5.3.34}$$

– $\sigma^2 \|\partial_t G^3\|_{H^{2n-3/2}(\Sigma)}$ **bound:**
We finally deal with the $\sigma^2 \|\partial_t G^3\|_{H^{2n-3/2}(\Sigma)}$ term. We bound this by

$$\begin{aligned} \sigma^2 \|\partial_t(\partial_1 \eta u_1)\|^2_{H^{2n-3/2}(\Sigma)} &\lesssim \sigma^2 \left(\|\partial_t \partial_1 \hat{\eta} u_1\|^2_{2n-1} + \|\partial_1 \hat{\eta} \partial_t u_1\|^2_{2n-1} \right) \\ &\lesssim \sigma^2 \left(\|\partial_t \partial_1 \hat{\eta} \nabla^{2n-1} u_1\|_0^2 + \|\partial_1 \hat{\eta} \partial_t \nabla^{2n-1} u_1\|_0^2 \right) \\ &\quad + \sum_{j=1}^{2n-1} \sigma^2 \left(\|\nabla^j \partial_t \partial_1 \hat{\eta} \nabla^{2n-1-j} u_1\|_0^2 + \|\nabla^j \partial_1 \hat{\eta} \partial_t \nabla^{2n-1-j} u_1\|_0^2 \right) \\ &\lesssim \sigma^2 \left(\|\partial_t \eta\|^2_{1/2} \|u\|^2_{2n-1} + \|\eta\|^2_{1/2} \|\partial_t u\|^2_{2n-1} \right) \\ &\quad + \sigma^2 \left(\|\partial_t \eta\|^2_{2n-1/2} \|u_1\|^2_{2n-2} + \|\eta\|^2_{2n-1/2} \|\partial_t u_1\|^2_{2n-2} \right) \\ &\lesssim \sigma^2 \mathcal{E}_n^0 \mathcal{D}_n^0 + \mathcal{E}_n^0 \mathcal{D}_n^\sigma \lesssim P(\sigma) \mathcal{E}_n^0 \mathcal{D}_n^\sigma. \end{aligned} \tag{5.3.35}$$

Bounding G^4: Let $j \in \{0, 1, \ldots, n-1\}$. Recall that

$$\begin{aligned} G^4 &= (pI - \mu \mathbb{D} u)(e_3 - \mathcal{N}) + (\mu \mathbb{D}_{\mathcal{A}-I} u) \mathcal{N} + \left(g\eta + A\omega^2 f''(\omega t) \eta \right)(e_3 - \mathcal{N}) \\ &\quad - (-\sigma \mathfrak{H}(\eta))(e_3 - \mathcal{N}) - (-\sigma(\Delta \eta - \mathfrak{H}(\eta))) e_3. \end{aligned} \tag{5.3.36}$$

We need to control $\partial_t^j G^4$ in $H^{2n-2j-1/2}(\Sigma)$ by $\mathcal{E}_n^0 \mathcal{D}_n^\sigma + \mathcal{K} \mathcal{F}_n$ and in $H^{2n-2j-3/2}(\Sigma)$ by $\mathcal{E}_n^0 \mathcal{E}_n^\sigma + \mathcal{K} \mathcal{F}_n$, and when $n = 1$ we can drop the $\mathcal{K} \mathcal{F}_n$ term. We do these term by term.

- **First term:** We handle $j = 0$ separately later, so let $1 \le j \le n-1$. We may immediately use Sobolev product estimates since Σ is two dimensional, so

$$\begin{aligned} \left\|\partial_t^j\left[(pI - \mu\mathbb{D}u)(e_3 - \mathcal{N})\right]\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 &\lesssim \sum_{\ell=0}^{j}\left\|\partial_t^\ell(pI - \mu\mathbb{D}u)\partial_t^{j-\ell}(e_3 - \mathcal{N})\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 \\ &\lesssim \sum_{\ell=0}^{j}\left\|\partial_t^\ell(pI - \mu\mathbb{D}u)\right\|_{H^{2n-2j-1/2}(\Sigma)}^2\left\|\partial_t^{j-\ell}\eta\right\|_{2n-2j+1/2}^2 \lesssim \mathcal{E}_n^0\mathcal{D}_n^\sigma. \end{aligned} \tag{5.3.37}$$

Note that a similar computation shows that if we only ask for $H^{2n-2j-3/2}(\Sigma)$, the we may indeed bound by $\mathcal{E}_n^0\mathcal{E}_n^\sigma$ since we don't need to use $\mathcal{D}_n^\sigma$ for the u term.

Now we handle $j = 0$. We have

$$\begin{aligned} &\|(pI - \mu\mathbb{D}u)(e_3 - \mathcal{N})\|_{H^{2n-1/2}(\Sigma)}^2 \\ &\quad\lesssim \|pI - \mu\mathbb{D}u\|_{L^\infty(\Sigma)}^2\|\nabla\eta\|_{2n-1/2}^2 + \|pI - \mu\mathbb{D}u\|_{H^{2n-1/2}(\Sigma)}^2\|\nabla\eta\|_{L^\infty(\Sigma)}^2. \end{aligned} \tag{5.3.38}$$

For the first term, we use $\mathcal{F}_n$ to bound $\|\nabla\eta\|_{2n-1/2}^2$ and we bound $\|pI\|_{L^\infty(\Sigma)}^2 \lesssim \|p\|_{H^3(\Sigma)}^2 \le \mathcal{K}$ and $\|\mu\mathbb{D}u\|_{L^\infty(\Sigma)}^2 \lesssim \|u\|_{H^3(\Sigma)}^2 \le \mathcal{K}$. Then, $\|\nabla\eta\|_{2n-1/2}^2 \lesssim \mathcal{F}_n$ so this first term is bounded as desired. The second term can be handled by

$$\|pI - \mu\mathbb{D}u\|_{H^{2n-1/2}(\Sigma)}^2\|\nabla\eta\|_{L^\infty(\Sigma)}^2 \lesssim \|pI - \mu\mathbb{D}u\|_{2n}^2\|\eta\|_3^2 \lesssim \mathcal{E}_n^0\min\left\{\mathcal{E}_n^0, \mathcal{D}_n^0\right\}. \tag{5.3.39}$$

- **Second term:** We again handle $j = 0$ separately and first work on $1 \le j \le n-1$. We easily have

$$\begin{aligned} \left\|\partial_t^j\left[\mu\mathbb{D}_{\mathcal{A}-I}u\mathcal{N}\right]\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 &\lesssim \left\|\partial_t^j\left[\left((\mathcal{A}-I)_{ik}\partial_k u_\ell + (\mathcal{A}-I)_{\ell k}\partial_k u_i\right)\mathcal{N}_\ell\right]\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 \\ &\lesssim \sum_{p+q+r=j}\left\|\left(\partial_t^p(\mathcal{A}-I)_{ik}\partial_t^q\partial_k u_\ell + \partial_t^p(\mathcal{A}-I)_{\ell k}\partial_t^q\partial_k u_i\right)\partial_t^r\mathcal{N}_\ell\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 \\ &\lesssim \sum_{p+q+r=j}\left\|\left(\partial_t^p(\mathcal{A}-I)_{ik}\partial_t^q\partial_k u_\ell + \partial_t^p(\mathcal{A}-I)_{\ell k}\partial_t^q\partial_k u_i\right)\right\|_{H^{2n-2j-1/2}(\Sigma)}^2\left\|\partial_t^r\mathcal{N}_\ell\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 \\ &\lesssim \mathcal{E}_n^0\mathcal{D}_n^\sigma. \end{aligned} \tag{5.3.40}$$

The $\mathcal{D}_n^\sigma$ can be replaced by $\mathcal{E}_n^\sigma$ when we only need to bound in $H^{2n-2j-3/2}(\Sigma)$ since the determining factor was the number of derivatives on u. We now work on $j = 0$. For this, we bound

$$\begin{aligned} \|\mu\mathbb{D}_{\mathcal{A}-I}u\mathcal{N}\|_{H^{2n-1/2}(\Sigma)} &\lesssim \|\mathbb{D}_{\mathcal{A}-I}u\|_{L^\infty(\Sigma)}\|\mathcal{N}\|_{H^{2n-1/2}(\Sigma)} + \|\mathbb{D}_{\mathcal{A}-I}u\|_{H^{2n-1/2}(\Sigma)}\|\mathcal{N}\|_{L^\infty(\Sigma)} \\ &\lesssim \mathcal{K}\mathcal{F}_n + \mathcal{E}_n^0\mathcal{D}_n^\sigma \end{aligned} \tag{5.3.41}$$

and we can replace $\mathcal{D}_n^\sigma$ with $\mathcal{E}_n^\sigma$ when we only need to bound in $H^{2n-2j-3/2}(\Sigma)$.

- **Third and fourth terms:** We omit details for these since they are similar to the first term. For the third, we may just use the L^∞ bound, while for the fourth the σ attached allows us to use our σ terms in the energy and dissipation.

- **Fifth term:** We recall some relevant computations from section 5.2. Note that the results done there apply directly when Sobolev product estimates apply, so we just need to worry about the case of bounding $n-1$ time derivatives of these term in $H^{2n-2(n-1)-3/2}(\Sigma) = H^{1/2}(\Sigma)$. In this case, we simply bound above by the norm in $H^{3/2}(\Sigma)$. Then, Sobolev product estimates apply and we bound

$$\left\| \partial_t^{n-1} \left[\left(\frac{1}{\sqrt{1+|\nabla\eta|^2}} \right) \Delta\eta \right] \right\|_{3/2}^2 \lesssim \sum_{\ell=0}^{n-1} \left\| \partial_t^\ell |\nabla\eta|^2 \right\|_{3/2}^2 \left\| \Delta \partial_t^{n-1-\ell} \eta \right\|_{3/2}^2 \lesssim \mathcal{E}_n^0 \mathcal{E}_n^0 \tag{5.3.42}$$

and

$$\begin{aligned} P(\sigma) \left\| \partial_t^{n-1} \frac{\partial_i \eta \langle \nabla\eta, \partial_i \nabla\eta \rangle}{(1+|\nabla\eta|^2)^{3/2}} \right\|_{3/2}^2 &\lesssim P(\sigma) \sum_{\ell=0}^{n-1} \left\| \partial_t^\ell (\partial_i \eta \langle \nabla\eta, \partial_i \nabla\eta \rangle) \right\|_{3/2}^2 \left\| \partial_t^{n-\ell} \frac{1}{(1+|\nabla\eta|^2)^{3/2}} \right\|_{3/2}^2 \\ &\lesssim P(\sigma) \mathcal{E}_n^0 \sum_{\ell=0}^{n-1} \left\| \sum_{m=0}^{n-1-\ell} \left(\partial_t^m \frac{1}{\sqrt{1+|\nabla\eta|^2}} \right) \left(\partial_t^{n-\ell-m} \frac{1}{1+|\nabla\eta|^2} \right) \right\|_{3/2}^2 \\ &\lesssim P(\sigma) \mathcal{E}_n^0 \mathcal{E}_n^0 \end{aligned} \tag{5.3.43}$$

as in section 5.2.

- $n=1$ **case**: It suffices to bound $\partial^\alpha G^4$ in $H^0(\Sigma)$ for $|\alpha| \le 1$. Note that the fifth term is already done by the above. The case for $\alpha = 0$, or the bound with $(\mathcal{E}_1^\sigma)^2$ is also easily done. Now we consider bounding $\|\partial^\alpha G^4\|_0$ for $\alpha = 1$ by $P(\sigma)\mathcal{E}_1^\sigma \mathcal{D}_1^\sigma$.

  For the first term, note that

$$\begin{aligned} \|\partial^\alpha (e_3 - \mathcal{N})(pI - \mu \mathbb{D}u)\|_0 &\lesssim \sum_{|\beta|+|\gamma| \le |\alpha|} \|\partial^\beta (e_3 - \mathcal{N}) \partial^\gamma (pI - \mu\mathbb{D}u)\|_0 \\ &\lesssim \sum_{|\beta|+|\gamma| \le |\alpha|} \|\partial^\beta (e_3 - \mathcal{N}) \partial^\gamma (pI - \mu\mathbb{D}u)\|_0 \\ &\lesssim \|\partial^\beta (\nabla\eta)\|_0 \|pI - \mu\mathbb{D}u\|_2 \\ &\lesssim \|\eta\|_2 \left(\|p\|_2 + \|\mathbb{D}u\|_2 \right) \lesssim \mathcal{E}_1^\sigma \mathcal{D}_1^\sigma. \end{aligned} \tag{5.3.44}$$

  Now to handle the second term, recall that

$$(\mathbb{D}_{I-\mathcal{A}}u)_{ij} = (I-\mathcal{A})_{ik} \partial_k u_j + (I-\mathcal{A})_{jk} \partial_k u_i = (I-\mathcal{A})_{i3} \partial_3 u_j + (I-\mathcal{A})_{j3} \partial_3 u_i. \tag{5.3.45}$$

  We claim that $\|\partial^\alpha (I-\mathcal{A})_{i3}\| \le \mathcal{E}_1^\sigma \mathcal{D}_1^\sigma$ for each $i = 1,2,3$. Once we achieve this, we will be done. Indeed, remark that

$$\begin{aligned} (\mathbb{D}_{\mathcal{A}-I} u \mathcal{N})_i &= (\mathbb{D}_{I-\mathcal{A}}u)_{ij} \mathcal{N}_j \\ &= (\mathbb{D}_{I-\mathcal{A}}u)_{i1}(-\partial_1 \eta) + (\mathbb{D}_{I-\mathcal{A}}u)_{i2}(\partial_2 \eta) + (\mathbb{D}_{I-\mathcal{A}}u)_{i3}. \end{aligned} \tag{5.3.46}$$

  By absorbing the $\partial_i \eta$ terms into $\mathcal{E}_1^\sigma \lesssim 1$, we establish the desired bound.

  We handle each of these three terms separately.

– **The A term:** Write

$$\|\partial^\alpha(AK\partial_3 u_1)\|_0 \lesssim \sum_{|\beta|+|\gamma|+|\delta|=1} \|\partial^\beta A \partial^\gamma K \partial^\delta(\partial_3 u_1)\|_0. \tag{5.3.47}$$

Upper bound $\|\partial^\gamma K\|_0 \lesssim \mathcal{E}_1^\sigma \lesssim 1$ as per usual, so that it remains to handle the A and $\partial_3 u_1$ parts. Upper bound the sum by

$$\sum \|\partial^\beta A \partial^\delta(\partial_3 u_1)\|_0 \lesssim \sum \|\partial^\beta(\partial_1 \eta)\|_0 \|\partial^\delta(\partial_3 u_1)\|_0 \lesssim \sum \|\eta\|_{|\beta|+1}\|u\|_{|\delta|+1}. \tag{5.3.48}$$

Now since $|\alpha| = 1$, every term is bounded by $\mathcal{E}_1^\sigma \mathcal{D}_1^\sigma$ so we get the G^2 component of eq. (5.3.4).

– **The B term:** Handled in exactly the same way.

– **The lone term:** Write

$$\|\partial^\alpha((1-K)\partial_3 u_3)\|_0 \lesssim \sum_{|\beta|+|\gamma|=1} \|\partial^\beta(1-K)\partial^\gamma(\partial_3 u_3)\|_0 \lesssim \sum_{|\beta|+|\gamma|=1} \|\eta\|_{|\beta|}\|u\|_{|\delta|+1}. \tag{5.3.49}$$

Now bound $\|\eta\|_{|\beta|} \lesssim \|\eta\|_1 \lesssim \mathcal{D}_1^\sigma$ and $\|u\|_{|\delta|+1} \lesssim \|u\|_2 \lesssim \mathcal{E}_1^\sigma$.

We now deal with the third term. Write

$$\begin{aligned}(g + A\omega^2 f''(\omega t)\eta) \cdot (\mathcal{N} - e_3) &= (g + A\omega^2 f''(\omega t)\eta) \cdot (-\partial_1 \eta, -\partial_2 \eta, 0) \\ &= -(g + A\omega^2 f''(\omega t))(\partial_1 \eta \eta_1 + \partial_2 \eta \eta_2).\end{aligned} \tag{5.3.50}$$

As a result,

$$\|\partial^\alpha(g + A\omega^2 f''(\omega t)\eta)(e_3 - \mathcal{N})\|_0 = \|(g + A\omega^2 f''(\omega t))(\partial_1 \eta \eta_1 + \partial_2 \eta \eta_2)\|_0 \lesssim \mathcal{E}_1^\sigma \mathcal{D}_1^\sigma. \tag{5.3.51}$$

To bound the fourth term, recall that

$$\mathfrak{H}(\eta) = \left(1 + |\nabla \eta|^2\right)^{-1/2} \Delta \eta - \left(1 + |\nabla \eta|^2\right)^{-3/2} \left(\partial_i \eta \left\langle \nabla \eta, \partial_i \nabla \eta \right\rangle\right). \tag{5.3.52}$$

First observe the following:

– All terms of the form $C(1 + |\nabla \eta|^2)^r$ may be upper bounded by $C \lesssim 1$.

– All normed terms associated with $\Delta \eta$ can be expressed solely in terms of $\|\Delta \partial^\alpha \eta\|$ for $|\alpha| \leq 1$; this is controlled by $\mathcal{E}_1^\sigma$.

– Every term in the expansion of $\partial_i \eta \left\langle \nabla \eta, \partial_i \nabla \eta \right\rangle$ can be written as the product of terms which take at most two derivatives of η (with exactly two occurring as a result of the $\partial_i \nabla \eta$ term). We can upper bound all remaining elements of the product by $\mathcal{E}_1^\sigma \lesssim 1$. Thus, when we take a derivative, this is the product of terms which have at most three derivatives of η, and thus is bounded above by a constant times $\|\eta\|_3 \lesssim \min\{\mathcal{E}_1^\sigma, \mathcal{D}_1^\sigma\}$.

As a result, $\|\mathfrak{H}(\eta)\|_0 \lesssim \mathcal{E}_1^\sigma$, and combined with $\|e_3 - \mathcal{N}\|_0 \lesssim \mathcal{D}_1^\sigma$ yields the desired bound. It remains to handle the $\sigma(\Delta \eta - \mathfrak{H}(\eta))e_3$ term. Write

$$\|\Delta \eta - \mathfrak{H}(\eta)\|_0 = \left\| \left((1 + |\nabla|\eta^2)^{-1/2} - 1\right) \Delta \eta + (1 + |\nabla|\eta^2)^{-3/2} \left\langle \nabla \eta, \partial_i \nabla \eta \right\rangle \right\|_0. \tag{5.3.53}$$

We already know how to handle the second term in the sum (the extra nonlinearities that we threw away before will be used but other than that everything is the same), so it remains to deal with the first term. We will case on whether we're taking zero or one derivatives.

– In the former case, note that rationalizing the numerator yields

$$\left\| \frac{1}{\sqrt{1+|\nabla\eta|^2}} - 1 \right\|_0 = \left\| \frac{|\nabla\eta|^2}{(1+\sqrt{1+|\nabla\eta|^2})\sqrt{1+|\nabla\eta|^2}} \right\|_0 \lesssim \|\nabla\eta\|_0^2. \tag{5.3.54}$$

Thus we may estimate the zero norm of the first term via $\||\nabla\eta|^2\|_0 \|\Delta\eta\|_0 \lesssim \mathcal{E}_\infty^\sigma \mathcal{D}_\infty^\sigma$.

– In the latter case, we replicate the analysis done before except replace temporal derivatives with spacial derivatives. In particular,

$$\begin{aligned} \left\| \partial_j \left[\left(\frac{1}{\sqrt{1+|\nabla\eta|^2}} - 1 \right) \Delta\eta \right] \right\|_0^2 &\lesssim \left\| \partial_j \frac{1}{\sqrt{1+|\nabla\eta|^2}} \right\|_0^2 \|\Delta\eta\|_0^2 + \left\| \left(\frac{1}{\sqrt{1+|\nabla\eta|^2}} - 1 \right) \right\|_0^2 \|\partial_j \Delta\eta\|_0^2 \\ &\lesssim \left\| \frac{\langle \nabla\eta, \partial_j \nabla\eta \rangle}{(1+|\nabla\eta|^2)^{3/2}} \right\|_0^2 \|\Delta\eta\|_0^2 + \left\| \frac{\sqrt{1+|\nabla\eta|^2} - 1}{\sqrt{1+|\nabla\eta|^2}} \right\|_0^2 \|\partial_j \Delta\eta\|_0^2 \\ &\lesssim \|\nabla\eta\|_0^2 \|\partial_j \nabla\eta\|_0^2 \|\Delta\eta\|_0^2 + \left\| |\nabla\eta|^2 \right\|_0^2 \|\partial_j \eta\|_0^2 \lesssim \mathcal{E}_1^\sigma \mathcal{D}_1^\sigma. \end{aligned} \tag{5.3.55}$$

□

5.4 Estimates on auxiliary terms

Our next result provides some bounds for nonlinearities appearing in integrals.

Proposition 5.4.1. *Let $\alpha \in \mathbb{N}^2$ with $|\alpha| = 2n$. Assume that $\mathcal{E}_n^\sigma \leq \delta$ for the universal $\delta \in (0,1)$ given by proposition 5.1.1. Then there exists a polynomial with nonnegative universal coefficients such that*

$$\left| \int_\Sigma \partial^\alpha \eta \partial^\alpha G^3 \right| \lesssim \sqrt{\mathcal{E}_n^0} \mathcal{D}_n^0 + \sqrt{\mathcal{D}_n^0 \mathcal{K} \mathcal{F}_n} \tag{5.4.1}$$

and

$$\left| \sigma \int_\Sigma \Delta \partial^\alpha \eta \partial^\alpha G^3 \right| \lesssim P(\sigma) \left(\sqrt{\mathcal{E}_n^0 \mathcal{D}_n^0 \mathcal{D}_n^\sigma} + \sqrt{\mathcal{D}_n^\sigma \mathcal{K} \mathcal{F}_n} \right). \tag{5.4.2}$$

Moreover, when $n = 1$ and $\sigma > 0$, we can improve the estimate above to

$$\left| \sigma \int_\Sigma \Delta \partial^\alpha \eta \partial^\alpha G^3 \right| \lesssim \frac{1+\sqrt{\sigma}}{\sigma} \sqrt{\mathcal{E}_1^\sigma} \mathcal{D}_1^\sigma. \tag{5.4.3}$$

Proof. We reproduce the proof from lemma 3.5 in [JTW16] for the sake of completeness.

We first have

$$\partial^\alpha G^3 = \partial^\alpha (\nabla\eta \cdot u) = \nabla\eta \cdot u + \sum_{0<\beta\leq\alpha} C_{\alpha\beta} \nabla \partial^{\alpha-\beta} \eta \partial^\beta u. \tag{5.4.4}$$

We now estimate

$$\sum_{0<\beta\le\alpha} C_{\alpha\beta}\left\|\nabla\partial^{\alpha-\beta}\eta\partial^\beta u\right\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{E}_n^0\mathcal{D}_n^0+\mathcal{K}\mathcal{F}_n \tag{5.4.5}$$

exactly as in previous nonlinearity bounds. Then,

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha\eta\left(\nabla\partial^{\alpha-\beta}\eta\partial^\beta u\right)\right| &\le \int_\Sigma (1+|\xi|^2)^{1/4}(1+|\xi|^2)^{-1/4}\left|\widehat{\partial^\alpha\eta}\right|\left|\nabla\widehat{\partial^{\alpha-\beta}\eta\partial^\beta} u\right| \\ &\le \|\partial^\alpha\eta\|_{-1/2}\|\nabla\partial^{\alpha-\beta}\eta\partial^\beta u\|_{1/2} \lesssim \sqrt{\mathcal{D}_n^0}\left(\sqrt{\mathcal{E}_n^0\mathcal{D}_n^0}+\sqrt{\mathcal{K}\mathcal{F}_n}\right)\end{aligned} \tag{5.4.6}$$

and

$$\begin{aligned}\left|\sigma\int_\Sigma \Delta\partial^\alpha\eta\left(\nabla\partial^{\alpha-\beta}\eta\partial^\beta u\right)\right| &\le \sigma\|\Delta\partial^\alpha\eta\|_{-1/2}\|\nabla\partial^{\alpha-\beta}\eta\partial^\beta u\|_{1/2} \\ &\lesssim \frac{\sigma}{\min\{1,\sigma\}}\sqrt{\min\{1,\sigma^2\}\|\eta\|_{2n+3/2}^2}\|\nabla\partial^{\alpha-\beta}\eta\partial^\beta u\|_{1/2} \\ &\lesssim P(\sigma)\sqrt{\mathcal{D}_n^\sigma}\left(\sqrt{\mathcal{E}_n^0\mathcal{D}_n^0}+\sqrt{\mathcal{K}\mathcal{F}_n}\right).\end{aligned} \tag{5.4.7}$$

For the first term, we use an integration by parts to see that

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha\eta\nabla\partial^\alpha\eta\cdot u\right| &= \left|\frac{1}{2}\int_\Sigma \nabla|\partial^\alpha\eta|\cdot u\right| = \left|\frac{1}{2}\int_\Sigma |\partial^\alpha\eta|^2(\partial_1 u_1+\partial_2 u_2)\right| \\ &\lesssim \|\partial^\alpha\eta\|_{-1/2}\|\partial^\alpha\eta\|_{1/2}\|\partial_1 u_1+\partial_2 u_2\|_{H^2(\Sigma)} \lesssim \sqrt{\mathcal{D}_n^0\mathcal{F}_n\mathcal{K}}\end{aligned} \tag{5.4.8}$$

and

$$\begin{aligned}\left|\int_\Sigma \sigma\Delta\partial^\alpha\eta\partial^\alpha\eta\cdot u\right| &= \left|\sum_{i=1}^2\int_\Sigma \sigma\partial_i\partial^\alpha\eta\partial_i(\nabla\partial^\alpha\eta\cdot u)\right| = \left|\sum_{i=1}^2\int_\Sigma \sigma\partial_i\partial^\alpha\eta(\nabla\partial^\alpha\partial_i\eta\cdot u+\nabla\partial^\alpha\eta\cdot\partial_i u)\right| \\ &= \left|\sum_{i=1}^2\int_\Sigma \sigma\partial_i\partial^\alpha\eta\nabla\partial^\alpha\partial_i\eta\cdot u+\sum_{i=1}^2\int_\Sigma \sigma\partial_i\partial^\alpha\eta(\nabla\partial^\alpha\eta\cdot\partial_i u)\right| \\ &= \left|\frac{1}{2}\int_\Sigma \sigma\nabla|\nabla\partial^\alpha\eta|^2\cdot u+\sum_{i=1}^2\int_\Sigma \sigma\partial_i\partial^\alpha\eta(\nabla\partial^\alpha\eta\cdot\partial_i u)\right| \\ &= \left|\frac{1}{2}\int_\Sigma \sigma|\nabla\partial^\alpha\eta|^2(\partial_1 u_1+\partial_2 u_2)+\sum_{i=1}^2\int_\Sigma \sigma\partial_i\partial^\alpha\eta(\nabla\partial^\alpha\eta\cdot\partial_i u)\right| \\ &\lesssim \sigma\|\nabla\partial^\alpha\eta\|_{-1/2}\|\nabla\partial^\alpha\eta\|_{1/2}\|\nabla u\|_{H^2(\Sigma)} \lesssim \sqrt{\mathcal{D}_n^\sigma\mathcal{F}_n\mathcal{K}}.\end{aligned} \tag{5.4.9}$$

For the $n=1$ and $\sigma>0$ case, we first estimate

$$\left|\int_\Sigma \sigma\Delta\partial^\alpha\eta\partial^\alpha G^3\right| \lesssim \sigma\|\Delta\partial^\alpha\eta\|_{-1/2}\|\partial^\alpha G^3\|_{1/2} \lesssim \sigma\|\eta\|_{7/2}\|\partial^\alpha G^3\|_{1/2} \lesssim \sqrt{\mathcal{D}_1^\sigma}\|\partial^\alpha G^3\|_{1/2}. \tag{5.4.10}$$

To conclude we use the definition of G^3 to bound

$$\left\|\partial^\alpha G^3\right\|_{1/2} \lesssim \|\eta\|_{5/2}\|u\|_3+\|u\|_2\|\eta\|_{7/2} \lesssim \frac{1+\sqrt{\sigma}}{\sigma}\sqrt{\mathcal{E}_1^\sigma\mathcal{D}_1^\sigma}. \tag{5.4.11}$$

□

We define the following auxiliary term which appear in later sections.

$$\mathcal{H}_n := \int_\Omega -\partial_t^{n-1} p F^{2,n} J + \frac{1}{2}|\partial_t^n u|^2 (J-1). \tag{5.4.12}$$

Proposition 5.4.2. *Let $\mathcal{H}_n$ be defined as in eq.* (5.4.12)*, and assume $\mathcal{E}_n^\sigma \leq \delta$ for the $\delta \in (0,1)$ given by proposition 5.1.1. Furthermore, suppose that $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$. Then*

$$|\mathcal{H}| \lesssim (\mathcal{E}_n^0)^{3/2}. \tag{5.4.13}$$

Proof. We can bound

$$|\mathcal{H}_n| \leq \|\partial_t^{n-1} p\|_0 \|F^{2,n}\|_0 \|J\|_{L^\infty} + \frac{1}{2}\|J-1\|_{L^\infty} \|\partial_t^n u\|_0^2. \tag{5.4.14}$$

Then we use proposition 5.1.1 and theorem 5.2.1 to estimate

$$\|F^{2,n}\|_0 \|J\|_{L^\infty} \lesssim \mathcal{E}_n^0. \tag{5.4.15}$$

Using the Sobolev embedding $H^3 \hookrightarrow C^1$,

$$\|J-1\|_{L^\infty} \lesssim \|\hat{\eta}\|_{C^1} \lesssim \|\hat{\eta}\|_{H^3} \lesssim \|\eta\|_{5/2} \lesssim \sqrt{\mathcal{E}_n^0}. \tag{5.4.16}$$

Therefore

$$|\mathcal{H}_n| \lesssim \sqrt{\mathcal{E}_n^0}\Big(\|\partial_t^{n-1} p\|_0 \sqrt{\mathcal{E}_n^0} + \|\partial_t^n u\|_0^2\Big) \lesssim (\mathcal{E}_n^0)^{3/2}, \tag{5.4.17}$$

as desired. □

Chapter 6

General a priori estimates

The purpose of this section is to present a priori estimates that are general in the sense that they are valid for both the problem with and without surface tension. The general estimates presented here will be specially adapted later to each problem to prove different sorts of results.

6.1 Energy-dissipation evolution estimates

Let $\alpha \in \mathbb{N}^{1+2}$, and write

$$\begin{aligned} \overline{\mathcal{E}}_\alpha &= \int_\Omega \frac{1}{2}|\partial^\alpha u|^2 + \int_\Sigma \frac{1}{2}|g\partial^\alpha \eta|^2 + \frac{\sigma}{2}|\nabla \partial^\alpha \eta|^2 \\ \overline{\mathcal{D}}_\alpha &= \int_\Omega \frac{1}{2}|\mathbb{D}\partial^\alpha u|^2 \end{aligned} \tag{6.1.1}$$

for the part of the energy and dissipation responsible for the α derivatives.

Our first result derives energy-dissipation estimates for the time derivative component of the energy and dissipation functionals.

Theorem 6.1.1. *Assume that $\mathcal{E}_n^\sigma \le \delta$ for the universal $\delta \in (0,1)$ given by proposition 5.1.1. Suppose further that $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$. Let $\alpha \in \mathbb{N}^{1+2}$ be given by $\alpha = (n,0,0)$, i.e. $\partial^\alpha = \partial_t^n$. Then for $\bar{\mathcal{E}}_\alpha^\sigma$ and $\mathcal{D}_\alpha^\sigma$ given by eq. (6.1.1), there exists a polynomial P with nonnegative universal constants such that we have the estimate*

$$\frac{d}{dt}(\overline{\mathcal{E}_\alpha^\sigma} + \mathcal{H}_n) + \overline{\mathcal{D}}_\alpha \lesssim \left(\sum_{\ell=2}^{n+2} A\omega^\ell\right)\mathcal{D}_n^0 + P(\sigma)\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma. \tag{6.1.2}$$

Proof. We apply proposition 4.1.1 with $(v,q,\zeta) = \partial_t^n(u,q,\eta)$ to get

$$\begin{aligned} &\frac{d}{dt}\left[\int_\Omega \frac{|\partial_t^n u|^2 J}{2} + \int_\Sigma \frac{\sigma|\nabla \partial_t^n \eta|^2}{2} + \frac{g|\partial_t^n \eta|^2}{2}\right] + \int_\Omega \mu \frac{|\mathbb{D}_\mathcal{A} \partial_t^n u|^2 J}{2} = \\ &\quad \int_\Omega J\left(\partial_t^n u \cdot F^{1,n} + \partial_t^n p \cdot F^{2,n}\right) + \int_\Sigma \left(-\sigma\Delta\partial_t^n \eta + g\partial_t^n \eta\right) F^{3,n} - \int_\Sigma F^{4,n}\cdot(\partial_t^n u) - F^{5,n}(\partial_t^n u)\cdot \mathcal{N}. \end{aligned} \tag{6.1.3}$$

Now we estimate the terms on the right hand side of (6.1.3). We easily bound the last term by

$$\left|\int_\Sigma F^{5,n}(\partial_t^n u)\cdot\mathcal{N}\right| = \left|\int_\Sigma \left(\sum_{\ell=0}^{n} C_{\ell,n} A\omega^{2+\ell} f^{(2+\ell)}(\omega t)\partial_t^{n-\ell}\eta\right)(\partial_t^n u)\cdot\mathcal{N}\right| \lesssim \left(\sum_{\ell=2}^{n+2} A\omega^\ell\right)\mathcal{D}_n^0. \tag{6.1.4}$$

To handle the pressure term we first rewrite

$$\int_\Omega \partial_t^n p J F^{2,n} = \frac{d}{dt}\int_\Omega \partial_t^{n-1} p J F^{2,n} - \int_\Omega \partial_t^{n-1} p \partial_t (J F^{2,n}). \tag{6.1.5}$$

We then use theorem 5.2.1 to estimate

$$\left| \int_\Omega \partial_t^{n-1} p \partial_t (J F^{2,n}) \right| \le \|\partial_t^{n-1} p\|_0 \|\partial_t (J F^{2,n})\|_0 \lesssim P(\sigma)\sqrt{\mathcal{D}_n^\sigma}\sqrt{\mathcal{E}_n^0 \mathcal{D}_n^\sigma} = P(\sigma)\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma. \tag{6.1.6}$$

Using theorem 5.2.1 and proposition 5.1.1 and trace theory, we get that

$$\left| \int_\Omega J \partial_t^n u \cdot F^{1,n} - \int_\Sigma F^{4,n} \cdot \partial_t^n u \right| \lesssim \|\partial_t^n u\|_1 (\|F^{1,n}\|_0 + \|F^{4,n}\|_0) \lesssim P(\sigma)\sqrt{\mathcal{D}_n^\sigma}\sqrt{\mathcal{E}_n^0 \mathcal{D}_n^\sigma} = P(\sigma)\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma. \tag{6.1.7}$$

For the rest of the terms, we again use theorem 5.2.1 and Sobolev embeddings to bound

$$\left| \int_\Sigma (-\sigma \Delta \partial_t^n \eta + g \partial_t^n \eta) F^{3,n} \right| \lesssim \|\partial_t^n \eta\|_2 \|F^{3,n}\|_0 \lesssim P(\sigma)\sqrt{\mathcal{D}_n^\sigma}\sqrt{\mathcal{E}_n^0 \mathcal{D}_n^\sigma} = P(\sigma)\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma. \tag{6.1.8}$$

Next we rewrite some of the terms on the left side of the equations. proposition 5.1.1 allows us to bound

$$\frac{1}{2}\int_\Omega |\mathbb{D}\partial_t^n u|^2 \le \int_\Omega \frac{1}{2}|\mathbb{D}_{\mathcal{A}} \partial_t^n u|^2 J + C\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma \tag{6.1.9}$$

and

$$\int_\Omega \frac{1}{2}|\partial_t^n u|^2 J = \int_\Omega \frac{1}{2}|\partial_t^n u|^2 + \int_\Omega \frac{1}{2}|\partial_t^n u|^2 (J-1). \tag{6.1.10}$$

The theorem follows by combining the above estimates and rearranging.

□

Our next result provides energy-dissipation estimates for all derivatives besides the highest order temporal ones.

Theorem 6.1.2. *Suppose that $\mathcal{E}_n^\sigma \le \delta$ for $\delta \in (0,1)$ given in proposition 5.1.1. Let $\alpha \in \mathbb{N}^{1+2}$ be such that $|\alpha| \le 2n$ and $\alpha_0 < n$. Suppose further that $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$. Then there exists a polynomial P with nonnegative universal coefficients such that*

$$\frac{d}{dt}\overline{\mathcal{E}_\alpha^\sigma} + \overline{\mathcal{D}_\alpha} \lesssim \left(\sum_{\ell=2}^{n+1} A\omega^\ell\right)\mathcal{D}_n^0 + P(\sigma)\left(\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma + \sqrt{\mathcal{D}_n^\sigma \mathcal{K}\mathcal{F}_n}\right). \tag{6.1.11}$$

Moreover, when $n = 1$ and $\sigma > 0$ is a fixed constant, we can improve this to

$$\frac{d}{dt}\overline{\mathcal{E}_\alpha^\sigma} + \overline{\mathcal{D}_\alpha} \lesssim (A\omega^2 + A\omega^3)\mathcal{D}_1^0 + \frac{P(\sigma)}{\sigma}\sqrt{\mathcal{E}_1^\sigma}\mathcal{D}_n^\sigma. \tag{6.1.12}$$

Proof. We begin by applying proposition 4.2.1 on $(v, q, \zeta) = \partial^\alpha(u, p, \eta)$ to see that

$$\begin{aligned} \frac{d}{dt}\overline{\mathcal{E}}_\alpha + \overline{\mathcal{D}}_\alpha = &- \int_\Sigma \partial^\alpha \left(A\omega^2 f''(\omega t)\eta\right)\partial^\alpha u_3 + \int_\Omega \partial^\alpha u \cdot \partial^\alpha G^1 + \partial^\alpha p \partial^\alpha G^2 \\ &+ \int_\Sigma (-\sigma\Delta\partial^\alpha \eta + g\partial^\alpha \eta)\partial^\alpha G^3 - \partial^\alpha G^4 \cdot \partial^\alpha u. \end{aligned} \tag{6.1.13}$$

We will now estimate all of the terms appearing on the right side of (6.1.13). The first term is easily bounded using the duality between $H^{1/2}(\Sigma)$ and $H^{-1/2}(\Sigma)$ and trace theory:

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha \left(A\omega^2 f''(\omega t)\eta\right)\partial^\alpha u_3\right| &\lesssim \left(\sum_{\ell=2}^{n+1} A\omega^\ell\right)\left(\sum_{j=0}^{n-1}\|\partial_t^j\eta\|_{2n-2j-1/2}\right)\left(\sum_{j=0}^{n-1}\|\partial^j u_3\|_{H^{2n-2j+1/2}(\Sigma)}\right)\\ &\lesssim \left(\sum_{\ell=2}^{n+1} A\omega^\ell\right)\sqrt{\mathcal{D}_n^0}\left(\sum_{j=0}^{n-1}\|\partial^j u\|_{2n-2j+1}\right) \lesssim \left(\sum_{\ell=2}^{n+1} A\omega^\ell\right)\mathcal{D}_n^0. \quad (6.1.14)\end{aligned}$$

In order to estimate the remaining terms on the right side of eq. (6.1.13) we will break to cases based on α.

Case 1 – Pure spatial derivatives of highest order: In this case we first consider $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| = 2n$ and $\alpha_0 = 0$, i.e. ∂^α is purely spatial derivatives of the highest order. Now write $\alpha = \beta + \gamma$ for $|\beta| = 1$. We then use integration by parts and Theorem 5.3.1 to bound the G^1 term via

$$\left|\int_\Omega \partial^\alpha u \cdot \partial^\alpha G^1\right| = \left|\int_\Omega \partial^{\alpha+\beta} u \cdot \partial^\gamma G^1\right| \lesssim \|u\|_{2n+1}\left\|G^1\right\|_{2n-1} \lesssim P(\sigma)\sqrt{\mathcal{D}_n^0}\sqrt{\mathcal{E}_n^0\mathcal{D}_n^\sigma + \mathcal{K}\mathcal{F}_n}. \quad (6.1.15)$$

To bound the G^2 term, compute

$$\left|\int_\Omega \partial^\alpha p \cdot \partial^\alpha G^2\right| \le \|\partial^\alpha p\|_0\left\|\partial^\alpha G^2\right\|_0 \lesssim P(\sigma)\sqrt{\mathcal{D}_n^0}\sqrt{\mathcal{E}_n^0\mathcal{D}_n^\sigma + \mathcal{K}\mathcal{F}_n}. \quad (6.1.16)$$

For the G^3 term, the $-\sigma\Delta\partial^\alpha\eta\partial^\alpha G^3$ and $g\partial^\alpha\eta$ terms are handled by Proposition 5.4.1. Finally, to bound the G^4 term, we have

$$\begin{aligned}\left|\int_\Sigma \partial^\alpha G^4 \cdot \partial^\alpha u\right| &= \left\|\partial^\alpha G^4\right\|_{H^{-1/2}(\Sigma)}\|\partial^\alpha u\|_{H^{1/2}(\Sigma)} \lesssim \left\|G^4\right\|_{H^{2n-1/2}(\Sigma)}\|u\|_{2n+1}\\ &\lesssim P(\sigma)\sqrt{\mathcal{D}_n^0}\sqrt{\mathcal{E}_n^0\mathcal{D}_n^\sigma + \mathcal{K}\mathcal{F}_n}.\end{aligned} \quad (6.1.17)$$

Combining the above estimates yields the desired bound for this case. For the $n = 1$ and $\sigma > 0$ case, we can apply the special cases of Theorem 5.3.1 and Proposition 5.4.1 and the same computations as above to deduce the result for G^1, G^2 and G^3, noting that

$$P(\sigma) + \frac{1+\sqrt{\sigma}}{\sigma} \lesssim \frac{P(\sigma)}{\sigma} \quad (6.1.18)$$

where P denotes different universal polynomials on each side of the inequality. For G^4 we can use the same method as for G^1 to get

$$\left|\int_\Sigma \partial^\alpha G^4 \cdot \partial^\alpha u\right| = \left|\int_\Sigma \partial^\gamma G^4 \cdot \partial^{\alpha+\beta} u\right| \lesssim \left\|G^4\right\|_1\|u\|_3 \lesssim \sqrt{\mathcal{E}_1^\sigma}\mathcal{D}_1^\sigma. \quad (6.1.19)$$

Case 2 – Everything else: We now consider the remaining cases, i.e. either $|\alpha| \le 2n-1$ or else $|\alpha| = 2n$ and $1 \le \alpha_0 \le n$. In this case, the G^1, G^2, G^4 terms may be handled with theorem 5.3.1. For the G^3 term, we directly compute

$$\begin{aligned}&\left|\int_\Sigma \left(-\sigma\Delta\partial^\alpha\eta + g\partial^\alpha\eta + \partial^\alpha\left(A\omega^2 f''(\omega t)\eta\right)\right)\partial^\alpha G^3\right|\\ &\quad\lesssim \left\|-\sigma\Delta\partial^\alpha\eta + g\partial^\alpha\eta + \partial^\alpha\left(A\omega^2 f''(\omega t)\eta\right)\right\|_0\left\|\partial^\alpha G^3\right\|\\ &\quad\lesssim P(\sigma)\sqrt{\mathcal{D}_n^\sigma}\sqrt{\mathcal{E}_n^0\mathcal{D}_n^\sigma + \mathcal{K}\mathcal{F}_n}.\end{aligned} \quad (6.1.20)$$

We may now combine the two cases to conclude the desired theorem. In the case of $n = 1$ and $\sigma > 0$, we can apply the special cases of theorem 5.3.1 in the above. $\square$

By combining theorems 6.1.1 and 6.1.2 we get the following synthesized result.

Theorem 6.1.3. *Suppose that $\mathcal{E}_n^\sigma \le \delta$ for $\delta \in (0,1)$ given by proposition 5.1.1. Suppose further that $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$. Then we have the estimate*

$$\frac{d}{dt}\left(\overline{\mathcal{E}_n^\sigma} + \mathcal{H}_n\right) + \overline{\mathcal{D}_n} \lesssim \left(\sum_{\ell=2}^{n+2} A\omega^\ell\right)\mathcal{D}_n^0 + P(\sigma)\left(\sqrt{\mathcal{E}_n^0}\mathcal{D}_n^\sigma + \sqrt{\mathcal{D}_n^\sigma \mathcal{K}\mathcal{F}_n}\right), \tag{6.1.21}$$

where $\mathcal{H}_n$ is defined as in eq. (5.4.12)*. Moreover, when $n = 1$ and $\sigma > 0$, we have*

$$\frac{d}{dt}\left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) + \overline{\mathcal{D}_1} \lesssim (A\omega^2 + A\omega^3)\mathcal{D}_1^0 + \frac{P(\sigma)}{\sigma}\sqrt{\overline{\mathcal{E}_1^\sigma}}\mathcal{D}_1^\sigma. \tag{6.1.22}$$

6.2 Comparison estimates

Our goal now is to show that the full energy and dissipation, $\mathcal{E}_n$ and $\mathcal{D}_n$, can be controlled by their horizontal counterparts $\overline{\mathcal{E}_n^\sigma}$ and $\overline{\mathcal{D}}_n$ up to some error terms that can be made small. We begin with the result for the dissipation.

Theorem 6.2.1. *Suppose that $\mathcal{E}_n^\sigma \le \delta$ for $\delta \in (0,1)$ given by proposition proposition 5.1.1. Let $\mathcal{Y}_n$ be as defined in eq.* (5.3.1)*. If $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$, then*

$$\mathcal{D}_n^\sigma \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n. \tag{6.2.1}$$

Proof. We divide the proof into several steps.

Step 1 – Application of Korn's inequality: Korn's inequality (see Lemma 2.3 of [Bea84]) tells us that

$$\sum_{\substack{\alpha \in \mathbb{N}^{1+2} \\ |\alpha| \le 2n}} \|\partial^\alpha u\|_1^2 \lesssim \overline{\mathcal{D}}_n. \tag{6.2.2}$$

Since ∂_1 and ∂_2 account for all the spatial differential operators on Σ, we deduce from standard trace estimates that

$$\sum_{j=0}^{n} \|\partial_t^j u\|_{H^{2n-2j+1/2}(\Sigma)} \lesssim \sum_{\substack{\alpha \in \mathbb{N}^{1+2} \\ |\alpha| \le 2n}} \|\partial^\alpha u\|_{H^{1/2}(\Sigma)}^2 \lesssim \overline{\mathcal{D}}_n. \tag{6.2.3}$$

Step 2 – Elliptic estimates for the Stokes problem: With eq. (6.2.3) in hand, we can now use the elliptic theory associated to the Stokes problem to gain control of the velocity field and the pressure. For $j = 0, 1, \ldots, n-1$ we have that $\partial_t^j(u, p, \eta)$ solve the PDE

$$\begin{cases} \operatorname{div} S\left(\partial_t^j u, \partial_t^j p\right) = \partial_t^j G^1 - \partial_t\left(\partial_t^j u\right) & \text{in } \Omega \\ \operatorname{div}\left(\partial_t^j u\right) = \partial_t^j G^2 & \text{in } \Omega \\ \partial_t^j u = \partial_t^j u\big|_\Sigma & \text{on } \Sigma \\ \partial_t^j u = 0 & \text{on } \Sigma_b \end{cases}. \tag{6.2.4}$$

We may then apply the Stokes problem elliptic regularity estimates in theorem theorem B.2.1 to bound

$$\|\partial_t^{n-1}u\|_3^2+\|\nabla\partial_t^{n-1}p\|_1^2 \lesssim \|\partial_t^n u\|_1^2+\|\partial_t^{n-1}u\|_{H^{5/2}(\Sigma)}^2+\|\partial_t^{n-1}G^1\|_1^2+\|\partial_t^{n-1}G^2\|_2^2 \lesssim \mathcal{Y}_n+\overline{\mathcal{D}}_n. \quad (6.2.5)$$

The control of $\partial_t^{n-1}u$ provided by this bound then allows us to control $\partial_t^{n-2}u$ in a similar manner. We thus proceed iteratively with theorem B.2.1 with $m = 2n-2j-1$, counting down from $n-1$ temporal derivatives to 0 temporal derivatives in order to deduce that

$$\sum_{j=0}^{n-1}\|\partial_t^j u\|_{2n-2j+1}^2 + \|\nabla\partial_t^j p\|_{2n-2j-1}^2 \lesssim P(\sigma)\left(\mathcal{Y}_n + \overline{\mathcal{D}}_n\right). \quad (6.2.6)$$

Step 3 – Free surface function estimates: Next we derive estimates for the free surface function. Consider the dynamic boundary condition on Σ to write

$$[(pI-\mu\mathbb{D}u)e_3]\cdot e_3 = \left[(-\sigma\Delta\eta + (g+A\omega^2 f''(\omega t))\eta)e_3 + G^4\right]\cdot e_3. \quad (6.2.7)$$

Now for $i=1,2$ and $j=0,1,\dots,n-1$, apply $\partial_i\partial_t^j$ to the above and rearrange to obtain

$$\begin{aligned} -\sigma\Delta\partial_i\partial_t^j\eta + (g+A\omega^2 f''(\omega t))\partial_i\partial_t^j\eta &= -\sum_{0<\ell\le j}\partial_t^\ell\left(A\omega^2 f''(\omega t)\right)\partial_i\partial^{j-\ell}\eta \\ &+ \left(\partial_i\partial_t^j p - 2\mu\partial_3\partial_i\partial_t^j u_3\right) - \partial_i\partial_t^j G^4\cdot e_3. \end{aligned} \quad (6.2.8)$$

We then use this in the capillary operator estimate count up from $j=0,1,\dots,n-1$ in theorem B.1.1 and employ eq. (6.2.5) to see that

$$\begin{aligned} \|\partial_i\partial_t^j\eta\|_{2n-2j-3/2}^2 + \sigma^2\|\partial_i\partial_t^j\eta\|_{2n-2j+1/2}^2 &\lesssim \left\|-\sum_{0<\ell\le j}\partial_t^\ell\left(A\omega^2 f''(\omega t)\right)\partial_i\partial^{j-\ell}\eta\right\|_{2n-2j-3/2}^2 \\ &+ \left\|\left(\partial_i\partial_t^j p - 2\mu\partial_3\partial_i\partial_t^j u_3\right) - \partial_i\partial_t^j G^4\cdot e_3\right\|_{H^{2n-2j-3/2}(\Sigma)}^2 \\ &\lesssim \sum_{\ell=0}^{j-1}\left\|\partial_i\partial^\ell\eta\right\|_{2n-2j-3/2}^2 + \left\|\nabla\partial_t^j p\right\|_{2n-2j-1}^2 + \left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \left\|\partial_t^j G^4\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 \lesssim \mathcal{Y}+\overline{\mathcal{D}}. \end{aligned} \quad (6.2.9)$$

Recall that η has zero integral over Σ, so by using Poincaré's inequality, we also obtain

$$\sum_{j=0}^{n-1}\left\|\partial_t^j\eta\right\|_{2n-2j-1/2}^2 + \sigma^2\left\|\partial_t^j\eta\right\|_{2n-2j+3/2}^2 \lesssim \sum_{j=0}^{n-1}\sum_{i=1}^{2}\left\|\partial_i\partial_t^j\eta\right\|_{2n-3/2}^2 + \sigma^2\left\|\partial_i\partial_t^j\eta\right\|_{2n+1/2}^2 \lesssim \mathcal{Y}+\overline{\mathcal{D}}. \quad (6.2.10)$$

Next we estimate $\partial_t^j\eta$ for $j=1,2,\dots,n+1$ by employing the kinematic boundary condition

$$\partial_t^{j+1}\eta = \partial_t^j u_3 + \partial_t^j G^3. \quad (6.2.11)$$

We first use this and eq. (6.2.10) to bound

$$\|\partial_t\eta\|_{2n-1}^2 \lesssim \|u_3\|_{H^{2n-1}(\Sigma)}^2 + \left\|G^3\right\|_{H^{2n-1}(\Sigma)}^2 \lesssim \|u\|_{2n-1/2}^2 + \mathcal{Y}_n \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n \quad (6.2.12)$$

and then multiply by σ^2 in order to derive the similar estimate

$$\begin{aligned}\sigma^2\|\partial_t\eta\|_{2n+1/2}^2 &\lesssim \sigma^2\|u_3\|_{H^{2n+1/2}(\Sigma)}^2 + \sigma^2\left\|G^3\right\|_{H^{2n+1/2}(\Sigma)}^2 \\ &\lesssim \|u\|_{2n+1}^2 + \sigma^2\left\|G^3\right\|_{H^{2n+1/2}(\Sigma)}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n.\end{aligned} \tag{6.2.13}$$

Next we use a similar argument to control $\partial_t^2\eta$:

$$\left\|\partial_t^2\eta\right\|_{2n-2}^2 \lesssim \|\partial_t u_3\|_{H^{2n-2}(\Sigma)}^2 + \left\|\partial_t G^3\right\|_{H^{2n-2}(\Sigma)}^2 \lesssim \|\partial_t u\|_{2n-3/2}^2 + \left\|\partial_t G^3\right\|_{H^{2n-2}(\Sigma)}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n \tag{6.2.14}$$

and

$$\begin{aligned}\sigma^2\left\|\partial_t^2\eta\right\|_{2n-3/2}^2 &\lesssim \sigma^2\|\partial_t u_3\|_{H^{2n-3/2}(\Sigma)}^2 + \sigma^2\left\|\partial_t G^3\right\|_{H^{2n-3/2}(\Sigma)}^2 \\ &\lesssim \|\partial_t u_3\|_{2n-1}^2 + \sigma^2\left\|\partial_t G^3\right\|_{H^{2n-3/2}(\Sigma)}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n.\end{aligned} \tag{6.2.15}$$

With control of $\partial_t^2\eta$ in hand we can iterate to obtain control of ∂_t^j for $j = 3, 4, \ldots, n+1$. This yields the estimate

$$\begin{aligned}\sum_{j=3}^{n+1}\left\|\partial_t^j\eta\right\|_{2n-2j+5/2}^2 &\lesssim \sum_{j=3}^{n+1}\left\|\partial_t^{j-1}u_3\right\|_{H^{2n-2j+5/2}(\Sigma)}^2 + \left\|\partial_t^{j-1}G^3\right\|_{H^{2n-2j+5/2}(\Sigma)}^2 \\ &= \sum_{j=2}^{n}\left\|\partial_t^j u\right\|_{2n-2j+1}^2 + \left\|\partial_t^j G^3\right\|_{H^{2n-2j+1/2}(\Sigma)}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n.\end{aligned} \tag{6.2.16}$$

Remark 6.2.1. The last surface function bound of [Tic18] (equation (5.41)) is not relevant for us since there is no e_1 component in our dynamic boundary condition.

Summing the above bounds then shows the following surface function estimate:

$$\begin{aligned}&\|\partial_t\eta\|_{2n-1}^2 + \sigma^2\|\partial_t\eta\|_{2n+1/2}^2 + \left\|\partial_t^2\eta\right\|_{2n-2}^2 + \sigma^2\left\|\partial_t^2\eta\right\|_{2n-3/2}^2 \\ &\quad + \sum_{j=0}^{n-1}\left(\left\|\partial_t^j\eta\right\|_{2n-2j-1/2}^2 + \sigma^2\left\|\partial_t^j\eta\right\|_{2n-2j+3/2}^2\right) + \sum_{j=3}^{n+1}\left\|\partial_t^j\eta\right\|_{2n-2j+5/2}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n.\end{aligned} \tag{6.2.17}$$

Step 4 – Improved pressure estimates We now return to eq. (6.2.7) with eq. (6.2.17) in hand in order to improve our estimates for the pressure. Applying ∂_t^j for $j = 0, 1, \ldots, n-1$ shows that

$$\partial_t^j p = -\sigma\Delta\partial_t^j\eta + g\partial_t^j\eta + \partial_t^j\left(A\omega^2 f''(\omega t)\eta\right) + 2\partial_3\partial_t^j u_3 + \partial_t^j G^4 \cdot e_3. \tag{6.2.18}$$

We then use this with eq. (6.2.10) to bound

$$\sum_{j=0}^{n-1}\left\|\partial_t^j p\right\|_{H^0(\Sigma)}^2 \lesssim \sum_{j=0}^{n-1}\|\partial_t^j\eta\|_0^2 + \sigma^2\|\Delta\partial_t^j\eta\|_2^2 + \left\|\partial_t^j u\right\|_2^2 + \left\|\partial_t^j G^4\right\|_{H^0(\Sigma)}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n. \tag{6.2.19}$$

Now by a Poincaré-type inequality,

$$\sum_{j=0}^{n-1}\left\|\partial_t^j p\right\|_0^2 \lesssim \sum_{j=0}^{n-1}\left\|\nabla\partial_t^j p\right\|_0^2 + \left\|\partial_t^j p\right\|_{H^0(\Sigma)}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n. \tag{6.2.20}$$

Hence

$$\sum_{j=0}^{n-1}\left\|\partial_t^j p\right\|_{2n-2j}^2 \lesssim \sum_{j=0}^{n-1}\left\|\partial_t^j p\right\|_0^2 + \left\|\nabla\partial_t^j p\right\|_{2n-2j-1}^2 \lesssim \mathcal{Y}_n + \overline{\mathcal{D}}_n. \tag{6.2.21}$$

Step 5 – Conclusion The estimate eq. (6.2.1) now follows by combining the above bounds. □

We now explore the counterpart for the energy.

Theorem 6.2.2. *Suppose that $\mathcal{E}_n^\sigma \leq \delta$ for $\delta \in (0,1)$ given by proposition proposition 5.1.1. Let $\mathcal{W}_n$ be as defined in eq. (5.3.2). If $\sum_{\ell=2}^{n+1} A\omega^\ell \lesssim 1$, then there exists a polynomial P with nonnegative universal coefficients such that*

$$\mathcal{E}_n^\sigma \lesssim P(\sigma)\left(\mathcal{W}_n + \overline{\mathcal{E}_n^\sigma}\right). \tag{6.2.22}$$

Proof. We divide the proof into several steps.

Step 1 – Initial free surface terms To begin, note that

$$\sum_{\substack{\alpha\in\mathbb{N}^{1+2}\\|\alpha|\leq 2n}} \|\partial^\alpha \eta\|_0^2 + \sigma\|\nabla\partial^\alpha\eta\|_0^2 \lesssim \sum_{j=0}^n \left\|\partial_t^j\eta\right\|_{2n-2j}^2 + \sigma\left\|\nabla\partial_t^j\eta\right\|_{2n-2j}^2. \tag{6.2.23}$$

Since $\partial_t^j\eta$ has zero integral, we can then use Poincaré's inequality to conclude that

$$\sum_{j=0}^n \left\|\partial_t^j\eta\right\|_{2n-2j}^2 + \sigma\left\|\partial_t^j\eta\right\|_{2n-2j+1}^2 \lesssim \sum_{j=0}^n \left\|\partial_t^j\eta\right\|_{2n-2j}^2 + \sigma\left\|\nabla\partial_t^j\eta\right\|_{2n-2j}^2 \lesssim \overline{\mathcal{E}_n^\sigma}. \tag{6.2.24}$$

Step 2 – Elliptic estimates Rewrite the flattened equations in eq. (2.3.10) as

$$\begin{cases} \nabla p - \mu\Delta u = G^1 - \partial_t u & \text{in } \Omega \\ \operatorname{div} u = G^2 & \text{in } \Omega \\ \partial_t\eta = u_3 + G^3 & \text{on } \Sigma \\ (pI - \mu\mathbb{D}u)e_3 = (-\sigma\Delta\eta + g\eta + G^5)e_3 + G^4 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \end{cases}. \tag{6.2.25}$$

Note that in particular $(\partial_t^j u, \partial_t^j p, \partial_t^j \eta)$ for $j = 1, 2, \ldots, n-1$ satisfy the PDE

$$\begin{cases} \nabla\partial_t^j p - \mu\Delta\partial_t^j u = \partial_t^j G^1 - \partial_t^{j+1} u & \text{in } \Omega \\ \operatorname{div} \partial_t^j u = \partial_t^j G^2 & \text{in } \Omega \\ \partial_t^{j+1}\eta = \partial_t^j u_3 + \partial_t^j G^3 & \text{on } \Sigma \\ (\partial_t^j pI - \mu\mathbb{D}\partial_t^j u)e_3 = (-\sigma\Delta\partial_t^j\eta + g\partial_t^j\eta + \partial_t^j G^5)e_3 + \partial_t^j G^4 & \text{on } \Sigma \\ v = 0 & \text{on } \Sigma_b \end{cases}. \tag{6.2.26}$$

We may appeal to the elliptic estimates for the Stokes problem with stress boundary conditions theorem B.3.1 to obtain

$$\begin{aligned} \|\partial_t^{n-1}u\|_2^2 + \|\partial_t^{n-1}p\|_1^2 &\lesssim \|\partial_t^{n-1}G^1 - \partial_t^n u\|_0^2 + \|\partial_t^{n-1}G^2\|_1^2 \\ &\quad + \|(-\sigma\Delta\partial_t^{n-1}\eta + g\partial_t^{n-1}\eta + \partial_t^{n-1}G^5)e_3 + \partial_t^{n-1}G^4\|_{1/2}^2 \\ &\lesssim \|\partial_t^{n-1}G^1\|_0^2 + \|\partial_t^n u\|_0^2 + \|\partial_t^{n-1}G^2\|_1^2 \\ &\quad + \|\partial_t^{n-1}\eta\|_{1/2}^2 + \sigma^2\|\partial_t^{n-1}\eta\|_{5/2}^2 + \|\partial_t^{n-1}G^5\|_{1/2}^2 + \|\partial_t^{n-1}G^4\|_{H^{1/2}(\Sigma)}^2. \end{aligned} \tag{6.2.27}$$

Remark that

$$\|\partial_t^{n-1} G^5\|_{1/2}^2 \leq \sum_{0\leq \ell \leq n-1} \|A\omega^{\ell+2} f^{(\ell+2)}(\omega t)\partial_t^{(n-1)-\ell}\eta\|_{1/2}^2 \lesssim \sum_{0\leq \ell\leq n-1} \|\partial_t^\ell \eta\|_{1/2}^2. \tag{6.2.28}$$

As a result, we have

$$\begin{aligned}\|\partial_t^{n-1}u\|_2^2 + \|\partial_t^{n-1}p\|_1^2 &\lesssim \|\partial_t^{n-1}G^1\|_0^2 + \|\partial_t^n u\|_0^2 + \|\partial_t^{n-1}G^2\|_1^2 \\ &+ \|\partial_t^{n-1}\eta\|_{1/2}^2 + \sigma^2\|\partial_t^{n-1}\eta\|_{5/2}^2 + \sum_{0\leq\ell\leq n-1}\|\partial_t^\ell\eta\|_{1/2}^2 + \|\partial_t^{n-1}G^4\|_{H^{1/2}(\Sigma)}^2 \lesssim P(\sigma)\left(\mathcal{W}_n + \overline{\mathcal{E}_n^\sigma}\right).\end{aligned} \tag{6.2.29}$$

We in turn may induct downward to get bounds on $\partial_t^j u$ and $\partial_t^j p$ for $j = n-2, \dots, 1, 0$. Thus

$$\begin{aligned}&\sum_{j=0}^{n-1}\|\partial_t^j u\|_{2n-2j}^2 + \|\partial_t^j p\|_{2n-2j-1}^2 \\ &\qquad\lesssim \overline{\mathcal{E}_n^\sigma} + \sum_{j=0}^{n-1}\left\|\partial_t^j G^1\right\|_{2n-2j-2}^2 + \left\|\partial_t^j G^2\right\|_{2n-2j-1}^2 + \left\|\partial_t^j G^4\right\|_{H^{2n-2j-3/2}(\Sigma)}^2 \lesssim P(\sigma)\left(\mathcal{W}_n + \overline{\mathcal{E}_n^\sigma}\right).\end{aligned} \tag{6.2.30}$$

Step 3 – Improved estimates for time derivatives of the free surface function With the estimates of eq. (6.2.30) in hand, we can improve the estimates for the time derivatives of the free surface function by employing the kinematic boundary condition

$$\partial_t^{j+1}\eta = \partial_t^j u_3 + \partial_t^j G^3 \tag{6.2.31}$$

for $j = 0, 1, \dots, n-1$. Using this, trace theory, eq. (6.2.24), and eq. (6.2.30) provides us with the estimate

$$\|\partial_t\eta\|_{2n-1/2}^2 \lesssim \|u\|_{2n}^2 + \|G^3\|_{H^{2n-1/2}(\Sigma)}^2 \lesssim \mathcal{W}_n + \overline{\mathcal{E}_n^\sigma}. \tag{6.2.32}$$

Remark 6.2.2. Again, this differs from the version in [Tic18] (equations (5.57) and (5.58)) in that our kinematic boundary condition doesn't make reference to η, so we don't need two different versions.

We then iterate this argument to control $\partial_t^j\eta$ for $j = 0, 1, \dots, n-1$. This yields the bound

$$\sum_{j=1}^{n}\left\|\partial_t^j\eta\right\|_{2n-2j+3/2}^2 \lesssim \sum_{j=0}^{n-1}\left\|\partial_t^j u\right\|_{2n-2j}^2 + \left\|\partial_t^j G^3\right\|_{H^{2n-2j-1/2}(\Sigma)}^2 \lesssim \mathcal{W}_n + \overline{\mathcal{E}_n^\sigma}. \tag{6.2.33}$$

Step 4 – Conclusion The estimate in eq. (6.2.22) now follows by combining the above bounds.
□

Chapter 7

Vanishing surface tension problem

In this section we complete the development of the a priori estimates for the vanishing surface tension problem and for the problem with zero surface tension. With these estimates in hand we then prove theorems 3.3.2 and 3.3.3, which establish the existence of global-in-time decaying solutions and study the limit as surface tension vanishes.

7.1 Preliminaries

Here we record a simple preliminary estimate that will be quite useful in the subsequent analysis.

Proposition 7.1.1. *For $N \geq 3$ we have that*

$$\mathcal{K} \lesssim \min \left\{ \mathcal{E}^0_{N+2}, \mathcal{D}^0_{N+2} \right\}, \qquad \mathcal{F}_{N+2} \lesssim \mathcal{E}^0_{2N}. \tag{7.1.1}$$

Proof. By Sobolev embeddings and trace theory, $\mathcal{K} \lesssim \|u\|^2_{7/2} + \|\eta\|^2_{5/2} \leq \|u\|^2_4 + \|\eta\|^2_4$ and hence $\mathcal{K} \lesssim \mathcal{E}^0_2 \leq \mathcal{E}^0_{N+2}$ and $\mathcal{K} \lesssim \mathcal{D}^0_2 \leq \mathcal{D}^0_{N+2}$. On the other hand, $\mathcal{F}_{N+2} = \|\eta\|^2_{2N+4+1/2} \leq \|\eta\|^2_{2N+5}$ and $2N+5 \leq 4N$ for $N \geq 3$, so $\mathcal{F}_{N+2} \leq \mathcal{E}^0_{2N}$. $\square$

7.2 Transport estimate

We now turn to the issue of establishing structured estimates of the highest derivatives of η by appealing to the kinematic transport equation. We begin by recording a general estimate for fractional derivatives of solutions to the transport equation, proved by Danchin [Dan05a]. Note that the result in [Dan05a] is stated for $\Sigma = \mathbb{R}^2$, but it can be readily extended to periodic Σ of the form we use: see for instance [Dan05b].

Lemma 7.2.1 (Proposition 2.1 of [Dan05a])**.** *Let ζ be a solution to*

$$\begin{cases} \partial_t \zeta + w \cdot D\zeta = g & \text{in } \Sigma \times (0,T) \\ \zeta(t=0) = \zeta_0 \end{cases}. \tag{7.2.1}$$

Then there exists a universal constant $C > 0$ such that for any $0 \leq s < 2$,

$$\sup_{0 \leq r \leq t} \|\zeta(r)\|_s \leq \exp\left(C \int_0^t \|Dw(r)\|_{3/2} \, dr \right) \left[\|\zeta_0\|_s + \int_0^t \|g(r)\|_s \, dr \right]. \tag{7.2.2}$$

Proof. Use $p = p_2 = 2$, $N = 2$, and $\sigma = s$ in proposition 2.1 of [Dan05a] along with the embedding $H^{3/2} \hookrightarrow B^1_{2,\infty} \cap L^\infty$. $\square$

We now parlay lemma 7.2.1 into the desired estimate for the highest spatial derivatives of η, $\mathcal{F}_{2N}$.

Theorem 7.2.1. *Assume that $\mathcal{E}_n^\sigma \le \delta$ for the universal $\delta \in (0,1)$ given by proposition 5.1.1. Then*

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \lesssim \exp\left(C\int_0^t \sqrt{\mathcal{K}(r)}\,dr\right)\left[\mathcal{F}_{2N}(0) + t\int_0^t (1+\mathcal{E}_{2N}^0)\mathcal{D}_{2N}^0\,dr + \left(\int_0^t \sqrt{\mathcal{K}\mathcal{F}_{2N}}\,dr\right)^2\right]. \tag{7.2.3}$$

Proof. We begin by introducing some notation. Throughout the proof we write $u = w + u_3 e_3$ for $w = u_1 e_1 + u_2 e_2$. We write D for the $2D$ gradient operator on Σ. Then η solves the transport equation

$$\partial_t \eta + w\cdot D\eta = u_3 \qquad \text{on } \Sigma. \tag{7.2.4}$$

We may then use lemma 7.2.1 with $s = 1/2$ to estimate

$$\sup_{0\le r\le t} \|\eta(r)\|_{1/2} \le \exp\left(C\int_0^t \|Dw(r)\|_{H^{3/2}(\Sigma)}\,dr\right)\left[\|\eta_0\|_{1/2} + \int_0^t \|u_3(r)\|_{H^{1/2}(\Sigma)}\,dr\right]. \tag{7.2.5}$$

We estimate the term in the exponential by

$$\|Dw\|_{H^{3/2}(\Sigma)} = \|\partial_1 u_1 + \partial_2 u_2\|_{H^{3/2}(\Sigma)} \lesssim \sqrt{\mathcal{K}} \tag{7.2.6}$$

and we may use trace theory to estimate $\|u_3(r)\|_{H^{1/2}(\Sigma)}^2 \lesssim \mathcal{D}_{2N}^0(r)$. This allows us to square both sides of eq. (7.2.5) and utilize Cauchy-Schwarz to deduce that

$$\begin{aligned}\sup_{0\le r\le t} \|\eta(r)\|_{1/2}^2 &\lesssim \exp\left(2C\int_0^t \sqrt{\mathcal{K}(r)}\,dr\right)\left[\|\eta_0\|_{1/2}^2 + \left(\int_0^t 1^2\,dr\right)\left(\int_0^t \mathcal{D}_n^0\,dr\right)\right] \\ &= \exp\left(2C\int_0^t \sqrt{\mathcal{K}(r)}\,dr\right)\left[\|\eta_0\|_{1/2}^2 + t\int_0^t \mathcal{D}_n^0\,dr\right].\end{aligned} \tag{7.2.7}$$

Next, we derive a higher regularity version of the estimate in eq. (7.2.7). To this end we choose any spatial multi-index $\alpha \in \mathbb{N}^2$ with $|\alpha| = 4N$, and we apply the operator ∂^α to eq. (7.2.4) to see that ∂^α solves the transport equation

$$\partial_t(\partial^\alpha \eta) + w\cdot D(\partial^\alpha \eta) = \partial^\alpha u_3 - \sum_{0<\beta\le\alpha} C_{\alpha,\beta}\partial^\beta w\cdot D\partial^{\alpha-\beta}\eta =: G^\alpha \tag{7.2.8}$$

with the initial condition $\partial^\alpha \eta_0$. We then again apply lemma 7.2.1 with $s = 1/2$ to find that

$$\sup_{0\le r\le t} \|\partial^\alpha \zeta(r)\|_{1/2} \le \exp\left(C\int_0^t \|Dw(r)\|_{H^{3/2}(\Sigma)}\,dr\right)\left[\|\partial^\alpha \eta_0\|_{1/2} + \int_0^t \|G^\alpha\|_{H^{1/2}(\Sigma)}\,dr\right]. \tag{7.2.9}$$

We will now estimate the G^α term above. For any spatial multi-index $\beta \in \mathbb{N}^2$ satisfying $2N+1 \le |\beta| \le 4N$, we use Sobolev product estimates in theorem C.1.1 with $s_1 = r = 1/2$ and $s_2 = 2$ and trace theory to bound

$$\left\|\partial^\beta w D\partial^{\alpha-\beta}\eta\right\|_{H^{1/2}(\Sigma)} \lesssim \left\|\partial^\beta w\right\|_{H^{1/2}(\Sigma)}\left\|D\partial^{\alpha-\beta}\eta\right\|_2 \lesssim \|u\|_{4N+1}\|\eta\|_{2N+2} \lesssim \sqrt{\mathcal{D}_{2N}^0\mathcal{E}_{2N}^0}. \tag{7.2.10}$$

On the other hand, for spatial multi-indices $\beta \in \mathbb{N}^2$ with $1 < |\beta| \leq 2N$, we instead bound

$$\left\|\partial^\beta w D\partial^{\alpha-\beta}\eta\right\|_{H^{1/2}(\Sigma)} \lesssim \left\|\partial^\beta w\right\|_{H^2(\Sigma)}\left\|D\partial^{\alpha-\beta}\eta\right\|_{1/2} \lesssim \|u\|_{2N+2}\|\eta\|_{4N-1/2} \lesssim \sqrt{\mathcal{D}^0_{2N}\mathcal{E}^0_{2N}}. \quad (7.2.11)$$

Finally, when $|\beta| = 1$, then we have

$$\left\|\partial^\beta w D\partial^{\alpha-\beta}\eta\right\|_{H^{1/2}(\Sigma)} \lesssim \left\|\partial^\beta w\right\|_{H^2(\Sigma)}\left\|D\partial^{\alpha-\beta}\eta\right\|_{1/2} \lesssim \|u\|_{H^3(\Sigma)}\|\eta\|_{4N+1/2} \lesssim \sqrt{\mathcal{K}\mathcal{F}_n}. \quad (7.2.12)$$

In order to bound G^α, it remains to bound $\partial^\alpha u_3$ which we obtain using trace theory:

$$\|\partial^\alpha u_3\|_{H^{1/2}(\Sigma)} \lesssim \|u\|_{4N+1} \lesssim \sqrt{\mathcal{D}^0_{2N}}. \quad (7.2.13)$$

On sum, we bound G^α by

$$\|G^\alpha\|_{H^{1/2}(\Sigma)} \lesssim \sqrt{\mathcal{D}^0_{2N}} + \sqrt{\mathcal{D}^0_{2N}\mathcal{E}^0_{2N}} + \sqrt{\mathcal{K}\mathcal{F}_{2N}}. \quad (7.2.14)$$

Returning to eq. (7.2.9), we square both sides and employ the bound from eq. (7.2.14) to find that

$$\begin{aligned} \sup_{0\leq r\leq t} \|\partial^\alpha\eta(r)\|^2_{1/2} \lesssim \exp&\left(2C\int_0^t \sqrt{\mathcal{K}}\, dr\right) \\ &\left[\|\partial^\alpha\eta\|^2_{1/2} + t\int_0^t (1+\mathcal{E}^0_{2N})\mathcal{D}^0_{2N}\, dr + \left(\int_0^t \sqrt{\mathcal{K}\mathcal{F}_{2N}}\, dr\right)^2\right]. \end{aligned} \quad (7.2.15)$$

Then the estimate in eq. (7.2.3) follows by summing eq. (7.2.15) over all $|\alpha| = 4N$, addint the resulting inequality to eq. (7.2.7), and using the fact that $\|\eta\|_{4N+1/2} \lesssim \|\eta\|_{1/2} + \sum_{|\alpha|=4N}\|\partial^\alpha\eta\|_{1/2}$. □

Next we show that if we know a prior that $\mathcal{G}_{2N}$ is small, then in fact it is possible to estimate $\mathcal{F}_{2N}$ more strongly than is done in theorem 7.2.1.

Theorem 7.2.2. *Let $\mathcal{G}^0_{2N}$ be defined by eq.* (3.3.2) *for $N \geq 3$. There exists a universal $\delta \in (0,1)$ such that if $\mathcal{G}^0_{2N}(T) \leq \delta$ and $\gamma \leq 1$, then*

$$\sup_{0\leq r\leq t} \mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}^0_{2N}(r)\, dr \quad (7.2.16)$$

for all $0 \leq t \leq T$.

Proof. According to proposition 7.1.1 and the assumed bounds, we may estimate

$$\int_0^t \sqrt{\mathcal{K}(r)}\, dr \lesssim \int_0^t \sqrt{\mathcal{E}^0_{N+2}(r)}\, dr \leq \sqrt{\delta}\int_0^\infty \frac{1}{(1+r)^{2N-4}}\, dr \lesssim \sqrt{\delta}. \quad (7.2.17)$$

Since $\delta \in (0,1)$, we thus have that for any universal $C > 0$

$$\exp\left(C\int_0^t \sqrt{\mathcal{K}(r)}\, dr\right) \lesssim 1. \quad (7.2.18)$$

Similarly,

$$\left(\int_0^t \sqrt{\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr\right)^2 \lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r)\right)\left(\int_0^t \sqrt{\mathcal{E}^0_{N+2}(r)}\,dr\right)^2 \lesssim \left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r)\right)\delta. \quad (7.2.19)$$

Then eqs. (7.2.17) to (7.2.19) and theorem 7.2.1 imply that

$$\sup_{0\le r\le t} \mathcal{F}_{2N}(r) \le C\left(\mathcal{F}_{2N}(0) + t\int_0^t \mathcal{D}^0_{2N}(r)\,dr\right) + C\delta\left(\sup_{0\le r\le t} \mathcal{F}_{2N}(r)\right) \quad (7.2.20)$$

for some $C > 0$. Then if δ is small enough so that $C\delta \le \frac{1}{2}$, we may absorb the right-hand $\mathcal{F}_{2N}$ term onto the left and deduce eq. (7.2.16). □

7.3 A priori estimates for $\mathcal{G}^0_{2N}$

Our goal now is to complete our a priori estimates for $\mathcal{G}^0_{2N}$. We start with the bounds of the high-tier terms and $\mathcal{F}_{2N}$.

Theorem 7.3.1. *There exist $\delta_0, \gamma_0 \in (0,1)$ such that if $0 \le \sigma \le 1$, $\mathcal{G}^0_{2N}(T) \le \delta_0$, and $\sum_{\ell=2}^{2N+2} A\omega^\ell \le \gamma_0$, then*

$$\sup_{0\le r\le t} \mathcal{E}^\sigma_{2N}(r) + \int_0^t \mathcal{D}^\sigma_{2N}(r)\ dr + \sup_{0\le r\le t} \frac{\mathcal{F}_{2N}(r)}{1+r} \lesssim \mathcal{E}^\sigma_{2N}(0) + \mathcal{F}_{2N}(0) \quad (7.3.1)$$

for all $0 \le t \le T$.

Proof. We first assume that δ_0 is small as in proposition 5.1.1, and small as in proposition 5.4.2 so that $|\mathcal{H}_{2N}| \lesssim (\mathcal{E}^0_{2N})^{3/2}$.

We invoke theorems 6.2.1 and 6.2.2 in order to bound

$$\mathcal{E}^\sigma_{2N} \lesssim \mathcal{W}_{2N} + \overline{\mathcal{E}^\sigma_{2N}} \qquad \text{and} \qquad \mathcal{D}^\sigma_{2N} \lesssim \mathcal{Y}_{2N} + \overline{\mathcal{D}_{2N}}. \quad (7.3.2)$$

According to theorem 5.3.1 we may then bound

$$\mathcal{W}_{2N} \lesssim \mathcal{E}^0_{2N}\mathcal{E}^\sigma_{2N} + \mathcal{K}\mathcal{F}_{2N} \qquad \text{and} \qquad \mathcal{Y}_{2N} \lesssim \mathcal{E}^0_{2N}\mathcal{D}^\sigma_{2N} + \mathcal{K}\mathcal{F}_{2N}. \quad (7.3.3)$$

Upon combining the above two equations with the given bound for $\mathcal{H}_{2N}$, we find that

$$\mathcal{E}^\sigma_{2N} \lesssim (\overline{\mathcal{E}^\sigma_{2N}} + \mathcal{H}_{2N}) + \mathcal{E}^0_{2N}\mathcal{E}^\sigma_{2N} + (\mathcal{E}^0_{2N})^{3/2} + \mathcal{K}\mathcal{F}_{2N} \quad \text{and} \quad \mathcal{D}^\sigma_{2N} \lesssim \overline{\mathcal{D}_{2N}} + \mathcal{E}^0_{2N}\mathcal{D}^\sigma_{2N} + \mathcal{K}\mathcal{F}_{2N}, \quad (7.3.4)$$

and consequently, if δ_0 is assumed to be small enough we may absorb the $\mathcal{E}^0_{2N}\mathcal{E}^\sigma_{2N} + (\mathcal{E}^0_{2N})^{3/2}$ and $\mathcal{E}^0_{2N}\mathcal{D}^\sigma_{2N}$ terms onto the left to arrive at the bounds

$$\mathcal{E}^\sigma_{2N} \lesssim (\overline{\mathcal{E}^\sigma_{2N}} + \mathcal{H}_{2N}) + \mathcal{K}\mathcal{F}_{2N} \qquad \text{and} \qquad \mathcal{D}^\sigma_{2N} \lesssim \overline{\mathcal{D}_{2N}} + \mathcal{K}\mathcal{F}_{2N}. \quad (7.3.5)$$

We apply theorem 6.1.3 with $n = 2N$ and integrate in time from 0 to t to see that

$$(\overline{\mathcal{E}^\sigma_{2N}}(t) + \mathcal{H}_{2N}(t)) + \int_0^t \overline{\mathcal{D}_{2N}}(r)\,dr \lesssim (\overline{\mathcal{E}^\sigma_{2N}}(0) + \mathcal{H}_{2N}(0)) + \left(\sum_{\ell=2}^{2N+2} A\omega^\ell\right)\int_0^t \mathcal{D}^0_{2N}(r)\,dr \\ + \int_0^t \sqrt{\mathcal{E}^0_{2N}(r)}\mathcal{D}^\sigma_{2N}(r)\,dr + \int_0^t \sqrt{\mathcal{D}^\sigma_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr. \quad (7.3.6)$$

We then combine this with the estimate in eq. (7.3.5) to arrive at the refined bound

$$\mathcal{E}^\sigma_{2N}(t) + \int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr \lesssim \mathcal{E}^\sigma_{2N}(0) + \left(\sum_{\ell=2}^{2N+2} A\omega^\ell\right) \int_0^t \mathcal{D}^0_{2N}(r)\,dr + \int_0^t \sqrt{\mathcal{E}^0_{2N}(r)}\mathcal{D}^\sigma_{2N}(r)\,dr \\ + \int_0^t \left(\mathcal{K}(r)\mathcal{F}_{2N}(r) + \sqrt{\mathcal{D}^\sigma_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\right) dr. \tag{7.3.7}$$

We now turn our attention to the $\mathcal{K}\mathcal{F}_{2N}$ terms appearing on the right side of eq. (7.3.7). To handle these we first note that $\mathcal{K} \lesssim \mathcal{E}^0_{N+2}$, as is shown in proposition 7.1.1. Thus

$$\mathcal{K}(r) \lesssim \mathcal{E}^0_{N+2}(r) = \frac{1}{(1+r)^{4N-8}}(1+r)^{4N-8}\mathcal{E}^0_{N+2} \lesssim \frac{1}{(1+r)^{4N-8}}\mathcal{G}_{2N}(T) \lesssim \frac{\delta_0}{(1+r)^{4N-8}}. \tag{7.3.8}$$

Next we use theorem 7.2.2 to see that for $0 \le r \le t$ we can estimate

$$\mathcal{F}_{2N}(r) \lesssim \mathcal{F}_{2N}(0) + (1+r)\int_0^r \mathcal{D}^0_{2N}(s)\,ds. \tag{7.3.9}$$

We may then combine eqs. (7.3.8) and (7.3.9) to estimate

$$\begin{aligned} \int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr &\lesssim \delta_0 \int_0^t \left(\frac{\mathcal{F}_{2N}(0)}{(1+r)^{4N-8}} + \frac{1}{(1+r)^{4N-7}}\int_0^r \mathcal{D}^0_{2N}(s)\,ds\right) \\ &\lesssim \delta_0\mathcal{F}_{2N}(0)\int_0^\infty \frac{dr}{(1+r)^{4N-8}} + \delta_0\left(\int_0^t \mathcal{D}^0_{2N}(r)\,dr\right)\left(\int_0^\infty \frac{dr}{(1+r)^{4N-7}}\right) \\ &\lesssim \delta_0\mathcal{F}_{2N}(0) + \delta_0\int_0^t \mathcal{D}^0_{2N}(r)\,dr, \end{aligned} \tag{7.3.10}$$

where here we have used $N \ge 3$ to guarantee that $(1+r)^{4N-8}$ and $(1+r)^{4N-7}$ are integrable on $(0,\infty)$. Similarly, we may estimate

$$\begin{aligned} \int_0^t \sqrt{\mathcal{D}^\sigma_{2N}(r)\mathcal{K}(r)\mathcal{F}_{2N}(r)}\,dr &\le \left(\int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr\right)^{1/2}\left(\int_0^t \mathcal{K}(r)\mathcal{F}_{2N}(r)\,dr\right)^{1/2} \\ &\lesssim \left(\int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr\right)^{1/2}\left(\delta_0\mathcal{F}_{2N}(0) + \delta_0\int_0^t \mathcal{D}^0_{2N}(r)\,dr\right)^{1/2} \\ &\lesssim \left(\mathcal{F}_{2N}(0) + \int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr\right)^{1/2}\left(\delta_0\mathcal{F}_{2N}(0) + \delta_0\int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr\right)^{1/2} \\ &\lesssim \sqrt{\delta_0}\mathcal{F}_{2N}(0) + \sqrt{\delta_0}\int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr. \end{aligned} \tag{7.3.11}$$

Now we plug eqs. (7.3.10) and (7.3.11) into eq. (7.3.7), bound $\mathcal{E}^0_{2N} \le \mathcal{G}^\sigma_{2N} \le \delta_0$, and use the fact that $\sqrt{\delta_0} \le \delta_0$ due to $\delta_0 < 1$ to arrive at the bound

$$\mathcal{E}^\sigma_{2N}(t) + \int_0^t \mathcal{D}^\sigma_{2N}(r)\,dr \lesssim \mathcal{E}_{2N}(0) + \mathcal{F}_{2N}(0) + \int_0^t \left(\sqrt{\delta} + \sum_{\ell=2}^{2N+2} A\omega^\ell\right)\mathcal{D}^\sigma_{2N}(r)\,dr. \tag{7.3.12}$$

Thus if $\gamma_0, \delta_0 \in (0,1)$ are chosen to be small enough, we may absorb the $\mathcal{D}^\sigma_{2N}(r)$ integral term onto the left to deduce that

$$\mathcal{E}^\sigma_{2N}(t) + \int_0^t \mathcal{D}^\sigma_{2N}(r)\, dr \lesssim \mathcal{E}^\sigma_{2N}(0) + \mathcal{F}_{2N}(0). \tag{7.3.13}$$

Upon combining eqs. (7.3.9) and (7.3.13) we deduce that the desired inequality holds. □

Next we establish the algebraic decay results for the low-tier energy.

Theorem 7.3.2. *There exists $\delta_0, \gamma_0 \in (0,1)$ such that if $0 \le \sigma \le 1$, $\sum_{\ell=2}^{N+4} A\omega^\ell \le \gamma_0$, and $\mathcal{G}^0_{2N}(T) \le \delta_0$, then*

$$\sup_{0\le r\le t} (1+r)^{4N-8} \mathcal{E}^\sigma_{N+2}(r) \lesssim \mathcal{E}^\sigma_{2N}(0) + \mathcal{F}_{2N}(0) \tag{7.3.14}$$

for all $0 \le t \le T$.

Proof. We prove in four tsteps.

Step 1 – Set up: Assume δ_0 is small as in proposition 5.1.1 and proposition 5.4.2. The latter allows us to estimate

$$|\mathcal{H}_{N+2}| \lesssim (\mathcal{E}^0_{N+2})^{3/2} \lesssim \sqrt{\mathcal{E}^0_{2N}} \mathcal{E}^0_{N+2} \tag{7.3.15}$$

since $N \ge 3$. Then by applying theorems 6.2.1 and 6.2.2 with $n = N+2$, together with theorem 5.3.1 and proposition 7.1.1 to get rid of the G nonlinearities and the $\mathcal{K}\mathcal{F}_{N+2}$ terms, we obtain the bounds

$$\begin{aligned} \mathcal{E}^\sigma_{N+2} &\lesssim \left(\overline{\mathcal{E}^\sigma_{N+2}} + \mathcal{H}_{N+2}\right) + \sqrt{\mathcal{E}^0_{2N}}\mathcal{E}^0_{N+2} + \mathcal{E}^0_{N+2}\mathcal{E}^\sigma_{N+2} + \mathcal{E}^0_{N+2}\mathcal{E}^0_{2N} \\ \mathcal{D}^\sigma_{N+2} &\lesssim \overline{\mathcal{D}_{N+2}} + \mathcal{E}^0_{N+2}\mathcal{D}^\sigma_{N+2} + \mathcal{E}^0_{2N}\mathcal{D}^0_{N+2} \end{aligned}. \tag{7.3.16}$$

Thus if we assume that δ_0 is small enough to absorb $\sqrt{\mathcal{E}^0_{2N}}\mathcal{E}^0_{N+2} + \mathcal{E}^0_{N+2}\mathcal{E}^\sigma_{N+2} + \mathcal{E}^0_{N+2}\mathcal{E}^0_{2N}$ and $\mathcal{E}^0_{N+2}\mathcal{D}^\sigma_{N+2} + \mathcal{E}^0_{2N}\mathcal{D}^0_{N+2}$ onto the left hand side, then we may arrive at the bounds

$$\mathcal{E}^\sigma_{N+2} \lesssim \left(\overline{\mathcal{E}^\sigma_{N+2}} + \mathcal{H}_{N+2}\right) \lesssim \mathcal{E}^\sigma_{N+2} \qquad \mathcal{D}^\sigma_{N+2} \lesssim \overline{\mathcal{D}_{N+2}} \lesssim \mathcal{D}^\sigma_{N+2}. \tag{7.3.17}$$

Step 2 – Interpolation estimates: Now set

$$\theta := \frac{4N-8}{4N-7} \in (0,1). \tag{7.3.18}$$

We claim that we have the interpolation estimate

$$\mathcal{E}_{N+2} \lesssim \left(\mathcal{D}^\sigma_{N+2}\right)^\theta \left(\mathcal{E}^\sigma_{2N}\right)^{1-\theta}. \tag{7.3.19}$$

For most of the terms appearing in $\mathcal{E}^\sigma_{N+2}$, this is a simple matter. Indeed, the definitions of $\mathcal{E}^\sigma_{2N}$ and $\mathcal{D}^\sigma_{N+2}$ and the assumption that $\sigma \le 1$ allow us to estimate

$$\begin{aligned} &\sum_{j=0}^{N+2} \|\partial_t^j u\|^2_{2(N+2)-2j} + \sum_{j=0}^{N+1} \|\partial_t^j p\|^2_{2(N+2)-2j-1} + \sigma\|\partial_t^j \eta\|^2_{2n-2j+1} + \sum_{j=2}^{N+2} \|\partial_t^j \eta\|_{2(N+2)-2j+3/2} + \sigma\|\eta\|^2_{2n+1} \\ &\qquad \lesssim \left(\mathcal{D}^\sigma_{N+2}\right)^\theta \left(\mathcal{E}^\sigma_{2N}\right)^{1-\theta} \end{aligned} \tag{7.3.20}$$

since the dissipation is actually coercive over the energy on these terms. To handle the remaning terms, we must use Sobolev interpolation. We begin with the most important term, which actually dictates the choice of θ. We have that

$$\left(2(N+2)-\frac{1}{2}\right)\theta+4N(1-\theta)=2(N+2) \iff \left(2N-\frac{7}{2}\right)\theta=2N-4 \tag{7.3.21}$$

so this θ is compatible with Sobolev norm estimates and so we obtain

$$\|\eta\|_{2(N+2)}^2 \leq \|\eta\|_{2(N+2)-1/2}^{2\theta}\|\eta\|_{4N}^{2(1-\theta)} \lesssim \left(\mathcal{D}_{N+2}^{\sigma}\right)^{\theta}\left(\mathcal{E}_{2N}^{\sigma}\right)^{1-\theta}. \tag{7.3.22}$$

Finally, we bound

$$\|\partial_t\eta\|_{2(N+2)-1/2}^2 \lesssim \|\partial_t\eta\|_{\theta(2(N+2)-1)+(1-\theta)(4N-1/2)}^2 \lesssim \|\partial_t\eta\|_{2(N+2)-1}^{2\theta}\|\partial_t\eta\|_{4N-1/2}^{2(1-\theta)} \lesssim \left(\mathcal{D}_{N+2}^{\sigma}\right)^{\theta}\left(\mathcal{E}_{2N}^{\sigma}\right)^{1-\theta} \tag{7.3.23}$$

and thus we have eq. (7.3.19) as claimed.

Step 3 – Differential inequality: Next we apply theorem 6.1.3 with $n=N+2$ in conjunction with proposition 7.1.1 to see that

$$\frac{d}{dt}\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_n\right)+\overline{\mathcal{D}_{N+2}} \lesssim \left(\sum_{\ell=2}^{N+4}A\omega^{\ell}\right)\mathcal{D}_{N+2}^{0}+\sqrt{\mathcal{E}_{N+2}^{0}}\mathcal{D}_{N+2}^{\sigma}+\sqrt{\mathcal{E}_{2N}^{0}}\mathcal{D}_{N+2}^{\sigma}. \tag{7.3.24}$$

We use this together with the bound $\mathcal{G}_{2N}^{\sigma}(T)\leq\delta_0$ and the dissipation bounds of eq. (7.3.17) to estimate

$$\frac{d}{dt}\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_n\right)+\overline{\mathcal{D}_{N+2}} \lesssim \left(\sqrt{\delta_0}+\sum_{\ell=2}^{N+4}A\omega^{\ell}\right)\overline{\mathcal{D}_{N+2}}. \tag{7.3.25}$$

Then by assuming that δ_0 and $\sum_{\ell=2}^{N+4}A\omega^{\ell}$ are small enough, we may absorb the $\overline{\mathcal{D}_{N+2}}$ onto the left of this inequality. Doing so and again invoking the dissipation bounds of eq. (7.3.17) gives us that

$$\frac{d}{dt}\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_{N+2}\right)+C_0\mathcal{D}_{N+2}^{\sigma}\leq 0 \tag{7.3.26}$$

for a universal constant $C_0>0$. We then use the energy estimate in eq. (7.3.17) to rewrite eq. (7.3.19) as

$$\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_{N+2}\right)^{1/\theta} \lesssim \mathcal{D}_{N+2}^{\sigma}\left(\mathcal{E}_{2N}^{\sigma}\right)^{(1-\theta)/\theta}. \tag{7.3.27}$$

We chain this together with the estimate in theorem 7.3.1 to write

$$\frac{C_1}{\mathcal{M}_0^s}\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_{N+2}\right)^{1+s}\leq\mathcal{D}_{N+2}^{\sigma} \tag{7.3.28}$$

for $C_1>0$ a universal constant, $s:=(1-\theta)/\theta=1/(4N-8)$, and $\mathcal{M}_0:=\mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0)$. Upon combining eqs. (7.3.26) and (7.3.28), we arrive at the differential inequality

$$\frac{d}{dt}\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_{N+2}\right)+\frac{C_0C_1}{\mathcal{M}_0^s}\left(\overline{\mathcal{E}_{N+2}^{\sigma}}+\mathcal{H}_{N+2}\right)^{1+s}\leq 0. \tag{7.3.29}$$

With eq. (7.3.29) in hand, we may integrate and argue as in the proofs of theorem 7.7 of [GT13] or proposition 8.4 of [JTW16] to deduce that

$$\sup_{0\leq r\leq t}(1+r)^{4N-8}\left(\overline{\mathcal{E}_{N+2}^{\sigma}(r)}+\mathcal{H}_{N+2}(r)\right)\lesssim\mathcal{M}_0=\mathcal{E}_{2N}(0)+\mathcal{F}_{2N}(0). \tag{7.3.30}$$

Then eq. (7.3.30) and the energy bound in eq. (7.3.17) yield eq. (7.3.14). □

As the final step in our a priori estimates for $\mathcal{G}^{\sigma}_{2N}$ we synthesize theorem 7.3.1 and theorem 7.3.2.

Theorem 7.3.3. *There exist* $\delta_0, \gamma_0 \in (0,1)$ *such that if* $0 \le \sum_{\ell=2}^{2N+2} A\omega^\ell < \gamma_0$*,* $0 \le \sigma \le 1$*, and* $\mathcal{G}_{2N}(T) \le \delta_0$*, then*

$$\mathcal{G}^{\sigma}_{2N}(T) \lesssim \mathcal{E}^{\sigma}_{2N}(0) + \mathcal{F}_{2N}(0). \tag{7.3.31}$$

Proof. We simply combine the estimates of theorems 7.3.1 and 7.3.2. □

7.4 Main results for the vanishing surface tension problem

Now that we have the a priori estimates of theorem 7.3.3 in hand, we may prove theorems 3.3.2 and 3.3.3 following previously developed arguments. For the sake of brevity we will omit full details and simply refer to the existing arguments.

Proof of theorem 3.3.2. The stated results follow by combining the local well-posedness theory, theorem 3.2.2, with the a priori estimates of theorem 7.3.3 and a continuation argument. The details of the continuation argument may be fully developed by following the arguments elaborated in theorem 1.3 of [GT13] or theorem 2.3 of [JTW16]. □

Proof of theorem 3.3.3. The results follow from the estimates of theorem 3.3.2 and standard compactness arguments. See theorem 1.2 of [TW14] or theorem 2.9 of [JTW16] for details. □

Chapter 8

Fixed surface tension problem

In this section we study the problem eq. (2.3.10) in the case of a fixed $\sigma > 0$. We develop a priori estimates and then present the proof of theorem 3.3.1. Although the structure of the proof is similar to that in [Tic18], this paper uses $n = 1$ to prove the main theorem rather than $n = 2$ as done in [Tic18]. This is because we wish to optimize our argument to give asymptotically better parameter regimes for A and ω; had we used $n = 2$, then we would have to require $\sum_{\ell=2}^{4} A\omega^\ell \lesssim 1$, which is worse than the regime in which $\sum_{\ell=2}^{3} A\omega^\ell \lesssim 1$ when we wish to consider large ω.

Note that in what follows in this section we break our convention of not allowing universal constants to depend on σ. All universal constants are allowed to depend on the fixed surface tension constant σ but are still not allowed to depend on A or ω.

8.1 A priori estimates for $\mathcal{S}_\lambda$

In order to prove theorem 3.3.1 we will introduce the following notation when $\lambda \in (0, \infty)$:

$$\mathcal{S}_\lambda(T) := \sup_{0 \le t \le T} e^{\lambda t} \mathcal{E}_1^\sigma(t) + \int_0^T \mathcal{D}_1^\sigma(t)\, dt. \tag{8.1.1}$$

We now develop the main a priori estimates with surface tension.

Theorem 8.1.1. *There exists $\delta_0, \gamma_0 \in (0, 1)$ such that if $\mathcal{S}_0(T) \le \delta_0$ and*

$$\left(A\omega^2 + A\omega^3\right) \le \gamma_0, \tag{8.1.2}$$

then there exists $\lambda = \lambda(\sigma) > 0$ such that

$$\mathcal{S}_\lambda(T) \lesssim \mathcal{E}_1^\sigma(0). \tag{8.1.3}$$

Proof. We assume that δ_0 is small enough that propositions 5.1.1 and 5.4.2 hold.

We use theorems 5.3.1, 6.2.1 and 6.2.2, as well as addition and subtraction of $\mathcal{H}$, to bound

$$\mathcal{E}_1^\sigma \lesssim \left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) + (\mathcal{E}_1^\sigma)^2 + (\mathcal{E}_1^\sigma)^{3/2} \qquad \text{and} \qquad \mathcal{D}_1^\sigma \lesssim \overline{\mathcal{D}_1} + \mathcal{E}_1^0 \mathcal{D}_1^\sigma. \tag{8.1.4}$$

By further restricting δ_0 we can use an absorbing argument to conclude

$$\mathcal{E}_1^\sigma + \mathcal{H}_1 \le \mathcal{E}_1^\sigma \lesssim \overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1 \quad \text{and} \quad \overline{\mathcal{D}_1} \le \mathcal{D}_1^\sigma \lesssim \overline{\mathcal{D}_1}. \tag{8.1.5}$$

We now employ theorem 6.1.3 with $n = 1$ (recalling that we now allow universal constants to depend on σ) and eq. (8.1.5) to get under appropriate smallness assumptions that

$$\frac{d}{dt}\left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) + \overline{\mathcal{D}_1} \lesssim \left(A\omega^2 + A\omega^3\right)\overline{\mathcal{D}_1} + \left(\sqrt{\overline{\mathcal{E}_1^\sigma}}\right)\overline{\mathcal{D}_1}, \tag{8.1.6}$$

We may then further restrict the size of δ_0 and γ_0 in order to absorb terms on the right onto the left. Note that this absorption requires A and ω to depend on σ. This yields the inequality

$$\frac{d}{dt}\left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) + \frac{1}{2}\overline{\mathcal{D}_1} \leq 0. \tag{8.1.7}$$

We defined $\mathcal{E}_1^\sigma$ and $\mathcal{D}_1^\sigma$ such that $\mathcal{E}_1^\sigma \lesssim \sigma^{-1}\mathcal{D}_1^\sigma$, so we can apply eq. (8.1.5) to get that there exists some $C > 0$ and $\lambda > 0$ depending on σ such that

$$\begin{aligned} \frac{1}{2}\overline{\mathcal{D}_1} &\geq \frac{2}{4C)}\mathcal{D}_1^\sigma \geq \frac{1}{4C}\mathcal{D}_1^\sigma + \frac{\sigma}{4C}\mathcal{E}_1^\sigma \\ &\geq \frac{1}{4C}\mathcal{D}_1^\sigma + \lambda\left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) \end{aligned} \tag{8.1.8}$$

Plugging this into eq. (8.1.7) gives

$$\frac{d}{dt}\left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) + \lambda\left(\overline{\mathcal{E}_1^\sigma} + \mathcal{H}_1\right) + \frac{1}{4C}\mathcal{D}_1^\sigma \leq 0. \tag{8.1.9}$$

We integrate this to get

$$e^{\lambda t}\left(\overline{\mathcal{E}_1^\sigma}(t) + \mathcal{H}_1(t)\right) + \frac{1}{4C}\int_0^t e^{\lambda r}\mathcal{D}_1^\sigma(r)\,dr \leq \left(\overline{\mathcal{E}_1^\sigma(0)} + \mathcal{H}_1(0)\right). \tag{8.1.10}$$

Now, appealing to eq. (8.1.5), we deduce

$$\sup_{0\leq t\leq T} e^{\lambda t}\mathcal{E}_1^\sigma(t) + \int_0^T \mathcal{D}_1^\sigma \lesssim \mathcal{E}_1^\sigma(0). \tag{8.1.11}$$

□

8.2 Proof of main result

Proof of theorem 3.3.1. We combine the local existence result in theorem 3.2.2 with the a priori estimates in theorem 8.1.1 and a continuation argument as in [GT13]. □

Appendix A

Auxilliary computations

A.1 Change of coordinates

Let $\Phi(y,t) = (y', y_3 - Af(\omega t), t) \in \mathbb{R}^3 \times \mathbb{R}^+$. Note that we may just pass in the spatial derivatives through the change of variables, so we just need to check $\partial_t \tilde{u}$ and $\tilde{u} \cdot \nabla \tilde{u}$. We compute

$$
\begin{aligned}
\partial_t \tilde{u}(y,t) &= \partial_t \left[u(\Phi(y,t)) + A\omega f'(\omega t) e_3 \right] \\
&= \begin{pmatrix} \partial_1 u(\Phi(y,t)) \\ \partial_2 u(\Phi(y,t)) \\ \partial_3 u(\Phi(y,t)) \\ \partial_t u(\Phi(y,t)) \end{pmatrix}^\top \begin{pmatrix} 0 \\ 0 \\ -A\omega f'(\omega t) \\ 1 \end{pmatrix} + A\omega^2 f''(\omega t) e_3 \\
&= -A\omega f'(\omega t) \partial_3 u(\Phi(y,t)) + \partial_t u(\Phi(y,t)) + A\omega^2 f''(\omega t) e_3
\end{aligned}
\tag{A.1.1}
$$

and

$$
\begin{aligned}
\tilde{u}(y,t) \cdot \nabla \tilde{u}(y,t) &= u(\Phi(y,t)) \cdot \nabla \left[u(\Phi(y,t)) \right] + A\omega f'(\omega t) e_3 \cdot \nabla \left[u(\Phi(y,t)) \right] \\
&= u(\Phi(y,t)) \cdot \nabla u(\Phi(y,t)) + A\omega f'(\omega t) \partial_3 u(\Phi(y,t)).
\end{aligned}
\tag{A.1.2}
$$

Note the cancellation of $A\omega f'(\omega t)\partial_3 u(\Phi(y,t))$ with these two terms, so the first equation becomes

$$
\partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta u + A\omega^2 f''(\omega t) e_3 = 0. \tag{A.1.3}
$$

The other equations are easy to track.

A.2 Flattening

If $f, \bar{f}$ are any functions with

$$
f(x,t) = \bar{f}(\Phi(x,t), t), \tag{A.2.1}
$$

then

$$
\begin{aligned}
\partial_t f(x,t) = \partial_t \left[\bar{f}(\Phi(x,t)) \right] &= \begin{pmatrix} \partial_1 \bar{f}(\Phi(y,t)) \\ \partial_2 \bar{f}(\Phi(y,t)) \\ \partial_3 \bar{f}(\Phi(y,t)) \\ \partial_t \bar{f}(\Phi(y,t)) \end{pmatrix}^\top \partial_t \begin{pmatrix} x_1 \\ x_2 \\ x_3 + \hat{\eta}(x,t)\left(1 + \frac{x_3}{b}\right) \\ t \end{pmatrix} \\
&= \partial_t \bar{f}(\Phi(y,t),t) + \partial_t \hat{\eta}(x,t) \left(1 + \frac{x_3}{b}\right) \partial_3 \bar{f}(\Phi(y,t),t).
\end{aligned}
\tag{A.2.2}
$$

Now note that

$$
\begin{aligned}
\partial_3 f(x,t) = \partial_3\left[\bar{f}(\Phi(x,t))\right] &= \begin{pmatrix} \partial_1\bar{f}(\Phi(y,t)) \\ \partial_2\bar{f}(\Phi(y,t)) \\ \partial_3\bar{f}(\Phi(y,t)) \\ \partial_t\bar{f}(\Phi(y,t)) \end{pmatrix}^{\top} \partial_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 + \hat{\eta}(x,t)\left(1+\frac{x_3}{b}\right) \\ t \end{pmatrix} \\
&= \partial_3\bar{f}(\Phi(y,t),t)\left(1 + \frac{\hat{\eta}(x,t)}{b} + \partial_3\hat{\eta}(x,t)\tilde{b}\right) = \partial_3\bar{f}(\Phi(y,t),t)J
\end{aligned}
\tag{A.2.3}
$$

so we find that

$$\partial_t f - \partial_t\hat{\eta}\tilde{b}K\partial_3 f = \partial_t\bar{f}\circ\Phi. \tag{A.2.4}$$

We also have that

$$\nabla f(x,t) = \nabla\left[\bar{f}(\Phi(x,t),t)\right] = \nabla\bar{f}(\Phi(x,t),t)\nabla\Phi(x,t) \tag{A.2.5}$$

so

$$\mathcal{A}\nabla f = \nabla\bar{f}\circ\Phi. \tag{A.2.6}$$

These computations can be used to directly to write the desired flattened equations.

A.3 Geometric energy-dissipation

Geometric identity We need to show that for each i, $\partial_k(J\mathcal{A}_{ik}) = 0$. For $i \in \{1,2\}$, this is true since

$$\partial_k(J\mathcal{A}_{ik}) = \partial_i J - \partial_3(\partial_i\hat{\eta}\tilde{b}) = \frac{\partial_i\hat{\eta}}{b} - \frac{\partial_i\hat{\eta}}{b} = 0 \tag{A.3.1}$$

and for $i = 3$, this is true since

$$\partial_k(J\mathcal{A}_{ik}) = \partial_3(1) = 0. \tag{A.3.2}$$

Then, we also get the identity

$$(u\cdot\nabla_{\mathcal{A}}v)\cdot vJ = (u_j\mathcal{A}_{jk}\partial_k v_i)v_i J = u_j J\mathcal{A}_{jk}\partial_k\left(\frac{v_i^2}{2}\right) + u_j\partial_k\left(J\mathcal{A}_{jk}\right)\frac{v_i^2}{2} = u_j\partial_k\left(J\mathcal{A}_{jk}\frac{|v|^2}{2}\right). \tag{A.3.3}$$

A.4 Flattened linearization

We linearize around the equilibrium solution $u = u_0 := 0, p = p_0 := 0, \eta = \eta_0 := 0$. Suppose that we have a one-parameter family of solutions to the problem eq. (2.3.10), $(-\varepsilon_0,\varepsilon) \ni \varepsilon \mapsto (u(\varepsilon), p(\varepsilon), \eta(\varepsilon))$ with $0 \mapsto (u_0, p_0, \eta_0)$. We then plug this family of solutions into the PDE, differentiate with respect to ε, and set $\varepsilon = 0$. We denote

$$u' := \frac{d}{d\varepsilon}u, \qquad \dot{u} := u'(\varepsilon = 0), \tag{A.4.1}$$

and similarly for p, η, and other quantities depending on the solution. We now differentiate eq. (2.3.10) with respect to ε. The first equation becomes

$$
\begin{aligned}
&\partial_t u' - \left(\partial_t\eta'\tilde{b}K\partial_3 u + \partial_t\eta\tilde{b}K'\partial_3 u + \partial_t\eta\tilde{b}K\partial_3 u'\right) \\
&\quad + (u'\cdot\nabla_{\mathcal{A}}u + u\cdot\nabla_{\mathcal{A}'}u + u\cdot\nabla_{\mathcal{A}}u') + (\operatorname{div}_{\mathcal{A}'} S_{\mathcal{A}} + \operatorname{div}_{\mathcal{A}} S'_{\mathcal{A}}) = 0,
\end{aligned}
\tag{A.4.2}
$$

the second equation becomes

$$\operatorname{div}_{\mathcal{A}'} u + \operatorname{div}_{\mathcal{A}} u' = 0, \tag{A.4.3}$$

the third equation becomes

$$\partial_t \eta' = u' \cdot \mathcal{N} + u \cdot \mathcal{N}', \tag{A.4.4}$$

the fourth equation becomes

$$\begin{aligned} S'_{\mathcal{A}}\mathcal{N} + S_{\mathcal{A}}\mathcal{N}' = \big(-\sigma\mathfrak{H}(\eta) + \big(g + A\omega^2 f''(\omega t)\big)\,\eta\big)\,\mathcal{N}' \\ + \big(-\sigma\mathfrak{H}(\eta)' + \big(g + A\omega^2 f''(\omega t)\big)\,\eta'\big)\,\mathcal{N}, \end{aligned} \tag{A.4.5}$$

and the fifth equation becomes

$$u' = 0. \tag{A.4.6}$$

Note that

$$\begin{aligned} \mathcal{A}' &= -\left((\nabla\Phi)^{-1}(\nabla\Phi)'(\nabla\Phi)^{-1}\right)^{\top} \\ S'_{\mathcal{A}} &= p'I - \mu\mathbb{D}_{\mathcal{A}'}u - \mu\mathbb{D}_{\mathcal{A}}u' \\ \mathcal{N}' &= (-\partial_1\eta', -\partial_2\eta', 0) \\ \mathfrak{H}(\eta)' &= \operatorname{div}\left(\left[\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right]'\right) = \sum_{i=1}^{2}\partial_i\left[\frac{\partial_i\eta}{\sqrt{1+|\nabla\eta|^2}}\right]' \\ &= \sum_{i=1}^{2}\partial_i\left[\frac{(1+|\nabla\eta|^2)^{1/2}\partial_i\eta' - \partial_i\eta(1+|\nabla\eta|^2)^{-1/2}\left\langle\nabla\eta,\nabla\eta'\right\rangle}{1+|\nabla\eta|^2}\right]. \end{aligned} \tag{A.4.7}$$

Now we set $\varepsilon = 0$, which gives us that

$$\mathcal{A}|_{\varepsilon=0} = I, \qquad \dot{\mathcal{A}} = -\nabla\dot{\Phi}^{\top} = -\left(0 \quad 0 \quad \nabla\left(\tilde{b}\dot{\eta}\right)\right), \qquad \dot{\mathfrak{H}(\eta)} = \sum_{i=1}^{2}\partial_i^2\dot{\eta} = \Delta\dot{\eta} \tag{A.4.8}$$

so we obtain

$$\begin{cases} \partial_t\dot{u} + \nabla\dot{p} - \mu\Delta\dot{u} = 0 & \text{in } \Omega \\ \operatorname{div}\dot{u} = 0 & \text{in } \Omega \\ \partial_t\dot{\eta} = \dot{u}_3 & \text{on } \Sigma \\ (\dot{p}I - \mu\mathbb{D}\dot{u})e_3 = (-\sigma\Delta\dot{\eta} + (g + A\omega^2 f''(\omega t))\dot{\eta})e_3 & \text{on } \Sigma \\ \dot{u} = 0 & \text{on } \Sigma_b \end{cases} \tag{A.4.9}$$

as desired.

A.5 Lemmas for bounding nonlinearities

A.5.1 Linearization of mean curvature

We prove a lemma used to bound $\mathfrak{H}(\eta) - \Delta\eta$. We wish to bound $\partial_t^\ell\sqrt{1+|\nabla\eta|^2}$. To do this, first note that by the Leibniz rule, we have

$$\partial_t^\ell|\nabla\eta|^2 = \partial_t^\ell(1+|\nabla\eta|^2) = \sum_{0\le m\le\ell}\binom{\ell}{m}\left(\partial_t^m\sqrt{1+|\nabla\eta|^2}\right)\left(\partial_t^{\ell-m}\sqrt{1+|\nabla\eta|^2}\right) \tag{A.5.1}$$

so

$$\partial_t^\ell \sqrt{1+|\nabla\eta|^2} = \partial_t^\ell |\nabla\eta|^2 - \frac{1}{2\sqrt{1+|\nabla\eta|^2}} \sum_{0<m<\ell} \binom{\ell}{m} \left(\partial_t^m \sqrt{1+|\nabla\eta|^2}\right) \left(\partial_t^{\ell-m} \sqrt{1+|\nabla\eta|^2}\right). \tag{A.5.2}$$

We now claim that for $\ell \geq 1$,

$$\left\| \partial_t^\ell \sqrt{1+|\nabla\eta|^2} \right\|_0^2 \lesssim \sum_{m=1}^{\ell} \left\| \partial_t^m |\nabla\eta|^2 \right\|_0^2. \tag{A.5.3}$$

For $\ell = 1$, we have

$$\left\| \partial_t \sqrt{1+|\nabla\eta|^2} \right\|_0^2 = \left\| \frac{\partial_t |\nabla\eta|^2}{\sqrt{1+|\nabla\eta|^2}} \right\|_0^2 \lesssim \left\| \partial_t |\nabla\eta|^2 \right\|_0^2 \tag{A.5.4}$$

and then using the formula above inductively, we find that

$$\begin{aligned} \left\| \partial_t^\ell \sqrt{1+|\nabla\eta|^2} \right\|_0^2 &\lesssim \left\| \partial_t^\ell |\nabla\eta|^2 \right\|_0^2 + \sum_{0<m<\ell} \left\| \partial_t^m \sqrt{1+|\nabla\eta|^2} \right\|_0^2 \left\| \partial_t^{\ell-m} \sqrt{1+|\nabla\eta|^2} \right\|_0^2 \\ &\lesssim \left\| \partial_t^\ell |\nabla\eta|^2 \right\|_0^2 + \sum_{0<m<\ell} \left\| \partial_t^m |\nabla\eta|^2 \right\|_0^2 = \sum_{m=1}^{\ell} \left\| \partial_t^m |\nabla\eta|^2 \right\|_0^2 \end{aligned} \tag{A.5.5}$$

so we conclude. Note that the $H^0(\Sigma)$ norm above may be replaced by $H^s(\Sigma)$ whenever Sobolev product estimates hold.

Appendix B

Elliptic estimates

Here we record basic elliptic estimates.

B.1 Capillary operator

Consider the problem

$$-\sigma\Delta\psi + g\psi = f \qquad \text{on } \mathbb{T}. \tag{B.1.1}$$

If $f \in H^{-1}(\mathbb{T}^n) = (H^1(\mathbb{T}^n))^*$, then a weak solution If $f \in H^{-1}(\mathbb{T}^n) = (H^1(\mathbb{T}^n))^*$, then a weak solution is readily found with a standard application of Riesz's representation theorem: there exists a unique $\psi \in H^1(\mathbb{T}^n)$ such that

$$\int_{\mathbb{T}^n} g\psi\varphi + \sigma\nabla\psi\cdot\nabla\varphi = \langle f, \varphi\rangle \tag{B.1.2}$$

Theorem B.1.1. *Let $s \geq 0$ and suppose that $f \in H^s(\mathbb{T}^n) \hookrightarrow H^{-1}(\mathbb{T}^n)$. Let $\psi \in H^1(\mathbb{T}^n)$ be the weak solution to eq.* (B.1.1)*. Then $\psi \in H^{s+1}(\mathbb{T}^n)$ and*

$$\|\psi\|_s \leq \tfrac{1}{g}\|f\|_s \qquad \textit{and} \qquad \|D^{2+s}\psi\|_0 \lesssim \tfrac{1}{\sigma}\|D^s f\|_0, \tag{B.1.3}$$

where $D = \sqrt{-\Delta}$. Moreover, if $\int_{\mathbb{T}^n}\psi = 0$, then

$$\|\psi\|_{s+2} \lesssim \tfrac{1}{\sigma}\|D^s f\|_0. \tag{B.1.4}$$

Proof. See Theorem A.1 of [Tic18]. □

B.2 Stokes operator with Dirichlet conditions

Consider the problem

$$\begin{cases} -\Delta u + \nabla p = f^1 & \text{in } \Omega \\ \operatorname{div} u = f^2 & \text{in } \Omega \\ u = f^3 & \text{on } \Sigma \\ u = 0 & \text{on } \Sigma_b \end{cases}. \tag{B.2.1}$$

The estimates for solutions are recorded in the following result, the proof of which is standard and can be found for instance in [Lad69].

Theorem B.2.1. *Let $m \in \mathbb{N}$. If $f^1 \in H^m(\Omega)$, $f^2 \in H^{m+1}(\Omega)$, and $f^3 \in H^{m+3/2}(\Sigma)$, the the solution pair (u, p) to eq.* (B.2.1) *satisfies $u \in H^{m+2}(\Omega)$, $\nabla p \in H^{m+1}(\Omega)$, and we have the estimate*

$$\|u\|_{m+2} + \|\nabla p\|_m \lesssim \|f^1\|_m + \|f^2\|_{m+2} + \|f\|_{m+3/2}. \tag{B.2.2}$$

B.3 Stokes operator with stress conditions

Consider the problem

$$\begin{cases} -\Delta u + \nabla p = f^1 & \text{in } \Omega \\ \operatorname{div} u = f^2 & \text{in } \Omega \\ u = 0 & \text{on } \Sigma_b \\ (pI - \mathbb{D}u)e_3 = f^3 & \text{on } \Sigma. \end{cases} \tag{B.3.1}$$

The estimates for solutions needed are recorded in the following result, the proof of which is standard and can be found for instance in [Bea81].

Theorem B.3.1. *Let $m \in \mathbb{N}$. If $f^1 \in H^m(\Omega)$, $f^2 \in H^{m+1}(\Omega)$, and $f^3 \in H^{m+1/2}(\Sigma)$, then the solution pair (u, p) to eq.* (B.3.1) *satisfies $u \in H^{m+2}(\Omega)$, $p \in H^{m+1}(\Omega)$, and we have the estimate*

$$\|u\|_{m+2} + \|p\|_{m+1} \lesssim \|f^1\|_m + \|f^2\|_{m+1} + \|f^3\|_{m+1/2}. \tag{B.3.2}$$

Appendix C

Analytic tools

C.1 Product estimates

In this section we record the necessary product estimates on Sobolev norms that we will need to get the correct bounds.

Theorem C.1.1. *The following hold on Σ and on Ω.*

1. *Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_1 > n/2$. Let $f \in H^{s_1}$, $g \in H^{s_2}$. Then $fg \in H^r$ and*
$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{C.1.1}$$
2. *Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{s_1}$, $g \in H^{s_2}$. Then $fg \in H^r$ and*
$$\|fg\|_{H^r} \lesssim \|f\|_{H^{s_1}} \|g\|_{H^{s_2}}. \tag{C.1.2}$$
3. *Let $0 \leq r \leq s_1 \leq s_2$ be such that $s_2 > r + n/2$. Let $f \in H^{-r}(\Sigma)$, $g \in H^{s_2}(\Sigma)$. Then $fg \in H^{-s_1}(\Sigma)$ and*
$$\|fg\|_{-s_1} \lesssim \|f\|_{-r} \|g\|_{s_2}. \tag{C.1.3}$$

Proof. See for example Lemma A.1 of [GT13]. □

C.2 Poisson extension

Suppose that $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$. We define the Poisson integral in $\Omega_- = \Sigma \times (-\infty, 0)$ by

$$\mathcal{P}f(x) := \sum_{n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})} e^{2\pi i n \cdot x'} e^{2\pi |n| x_3} \hat{f}(n), \tag{C.2.1}$$

where for $n \in (L_1^{-1}\mathbb{Z}) \times (L_2^{-1}\mathbb{Z})$ we have written

$$\hat{f}(n) := \int_\Sigma f(x') \frac{e^{-2\pi i n \cdot x'}}{L_1 L_2} \, dx'. \tag{C.2.2}$$

It is well-known that $\mathcal{P} : H^s(\Sigma) \to H^{s+1/2}(\Omega_-)$ is a bounded linear operator for $s > 0$. We now show that derivatives of $\mathcal{P}f$ can be estimated in the smaller domain Ω.

Lemma C.2.1. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is either in $\dot{H}^q(\Sigma)$ or $\dot{H}^{q-1/2}(\Sigma)$ for $q \in \mathbb{N}$. Then*

$$\|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|_{\dot{H}^{q-1/2}(\Sigma)}^2 \qquad \textit{and} \qquad \|\nabla^q \mathcal{P} f\|_0^2 \lesssim \|f\|_{\dot{H}^{q}(\Sigma)}^2. \tag{C.2.3}$$

Proof. See lemma A.3 in [GT13]. □

We will also need L^∞ estimates.

Lemma C.2.2. *Let $\mathcal{P}f$ be the Poisson integral of a function f that is in $\dot{H}^{q+s}(\Sigma)$ for $q \geq 1$ an integer and $s > 1$. Then*

$$\|\nabla^q \mathcal{P} f\|_{L^\infty}^2 \lesssim \|f\|_{\dot{H}^{q+s}}^2. \tag{C.2.4}$$

The same estimate holds for $q = 0$ if f satisfies $\int_\Sigma f = 0$.

Proof. See lemma A.4 in [GT13]. □

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